Binomial Heaps

Where We're Going

- ***Binomial Heaps (Today)***
  - A simple, flexible, and versatile priority queue.
- ***Lazy Binomial Heaps (Today)***
  - A powerful building block for designing more advanced data structures.
- ***Fibonacci Heaps (Tuesday)***
  - A famous and theoretically excellent priority queue.

Review: Priority Queues

Priority Queues

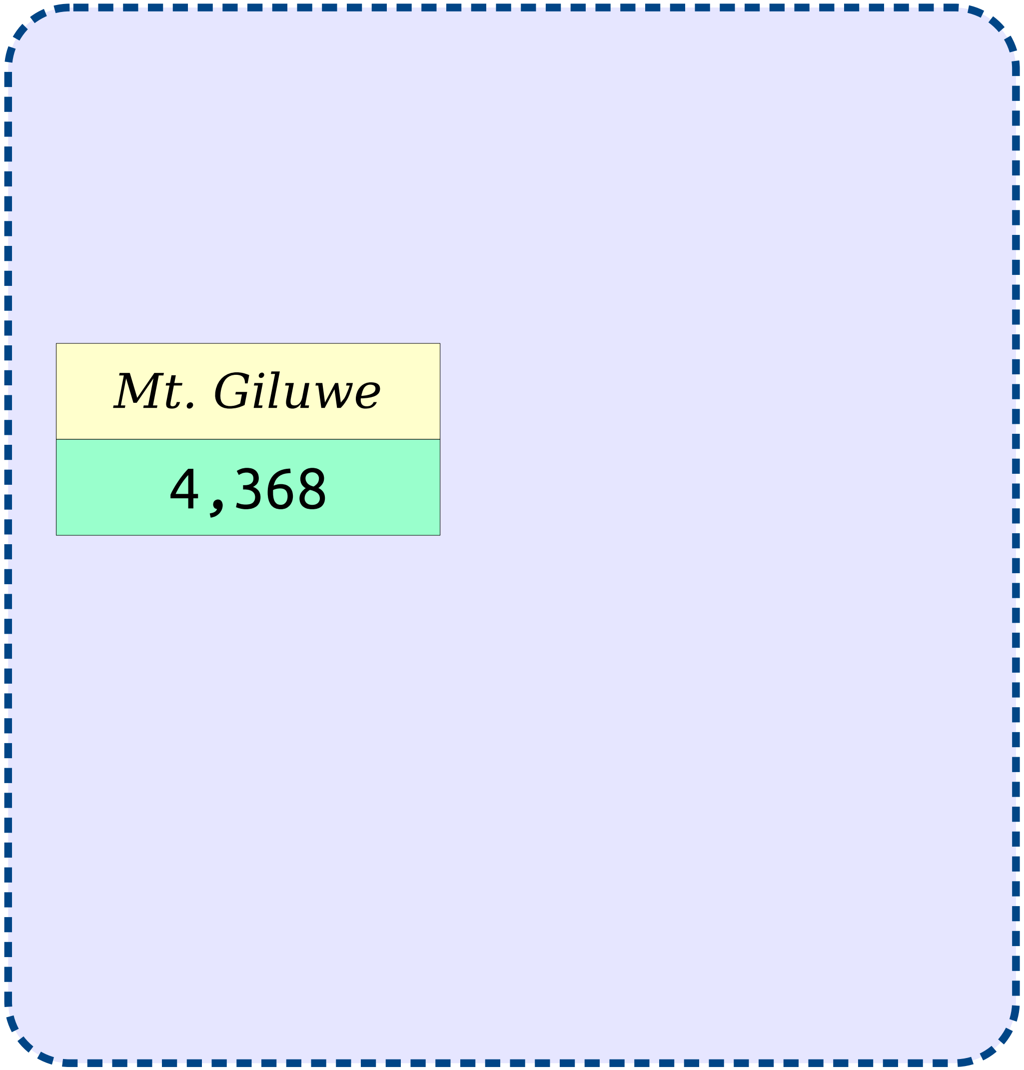
- A ***priority queue*** is a data structure that supports these operations:
  - *pq*.***enqueue***(*v*, *k*), which enqueues element *v* with key *k*;
  - *pq*.***find-min***(), which returns the element with the least key; and
  - *pq*.***extract-min***(), which removes and returns the element with the least key.
- They're useful as building blocks in a *bunch* of algorithms.

Priority Queues

- A ***priority queue*** is a data structure that supports these operations:
  - *pq*.***enqueue***(*v*, *k*), which enqueues element *v* with key *k*;
  - *pq*.***find-min***(), which returns the element with the least key; and
  - *pq*.***extract-min***(), which removes and returns the element with the least key.
- They're useful as building blocks in a *bunch* of algorithms.

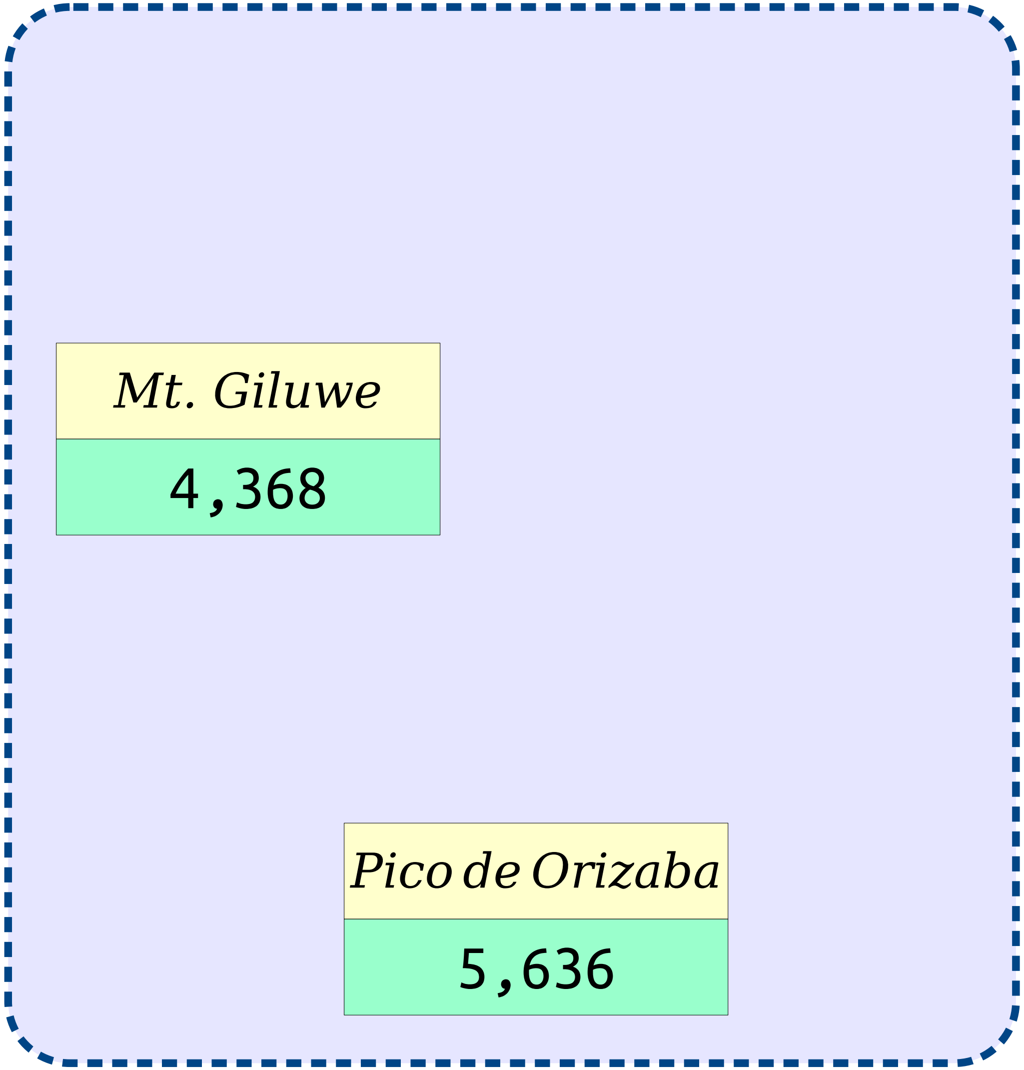
Priority Queues

- A ***priority queue*** is a data structure that supports these operations:
  - *pq*.***enqueue***(*v*, *k*), which enqueues element *v* with key *k*;
  - *pq*.***find-min***(), which returns the element with the least key; and
  - *pq*.***extract-min***(), which removes and returns the element with the least key.
- They're useful as building blocks in a *bunch* of algorithms.



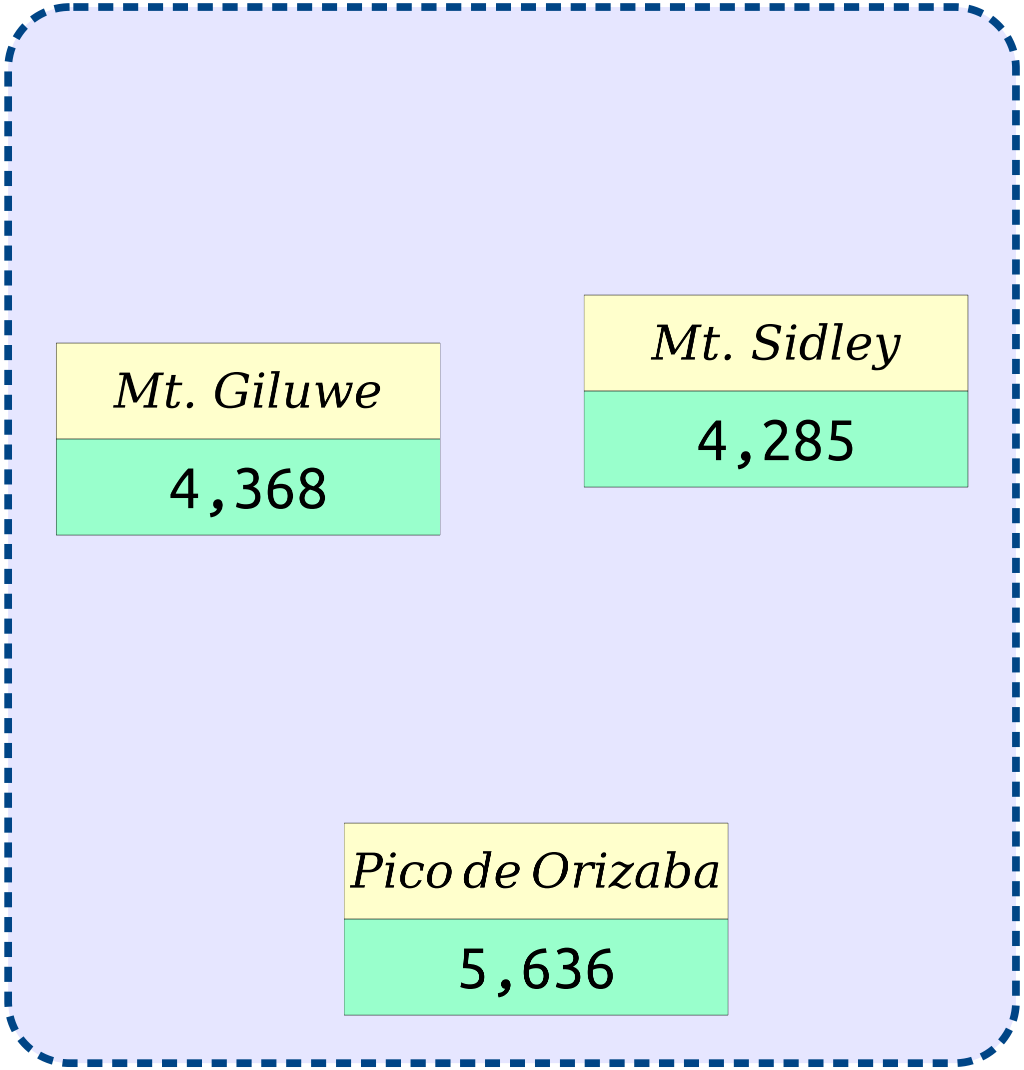
Priority Queues

- A ***priority queue*** is a data structure that supports these operations:
  - *pq*.***enqueue***(*v*, *k*), which enqueues element *v* with key *k*;
  - *pq*.***find-min***(), which returns the element with the least key; and
  - *pq*.***extract-min***(), which removes and returns the element with the least key.
- They're useful as building blocks in a *bunch* of algorithms.



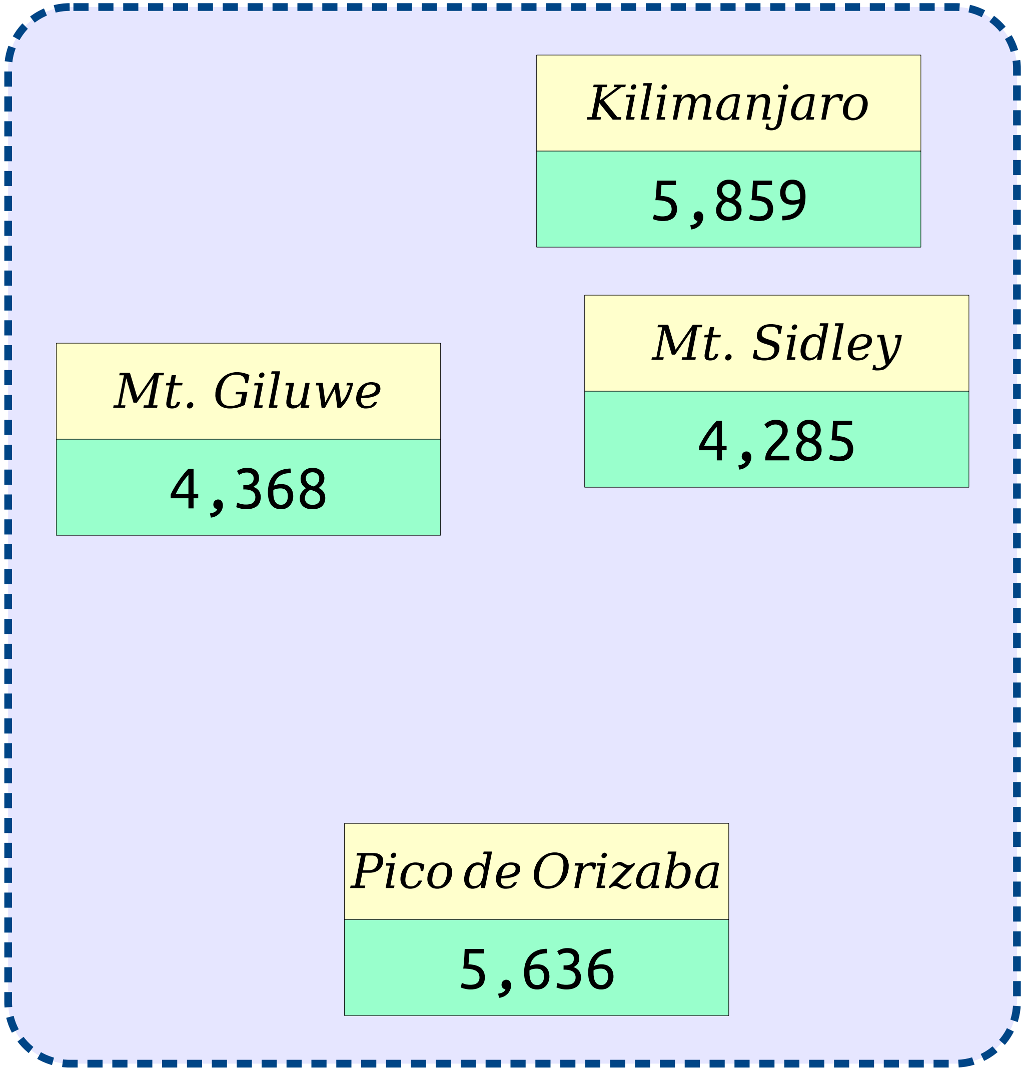
Priority Queues

- A ***priority queue*** is a data structure that supports these operations:
  - *pq*.***enqueue***(*v*, *k*), which enqueues element *v* with key *k*;
  - *pq*.***find-min***(), which returns the element with the least key; and
  - *pq*.***extract-min***(), which removes and returns the element with the least key.
- They're useful as building blocks in a *bunch* of algorithms.



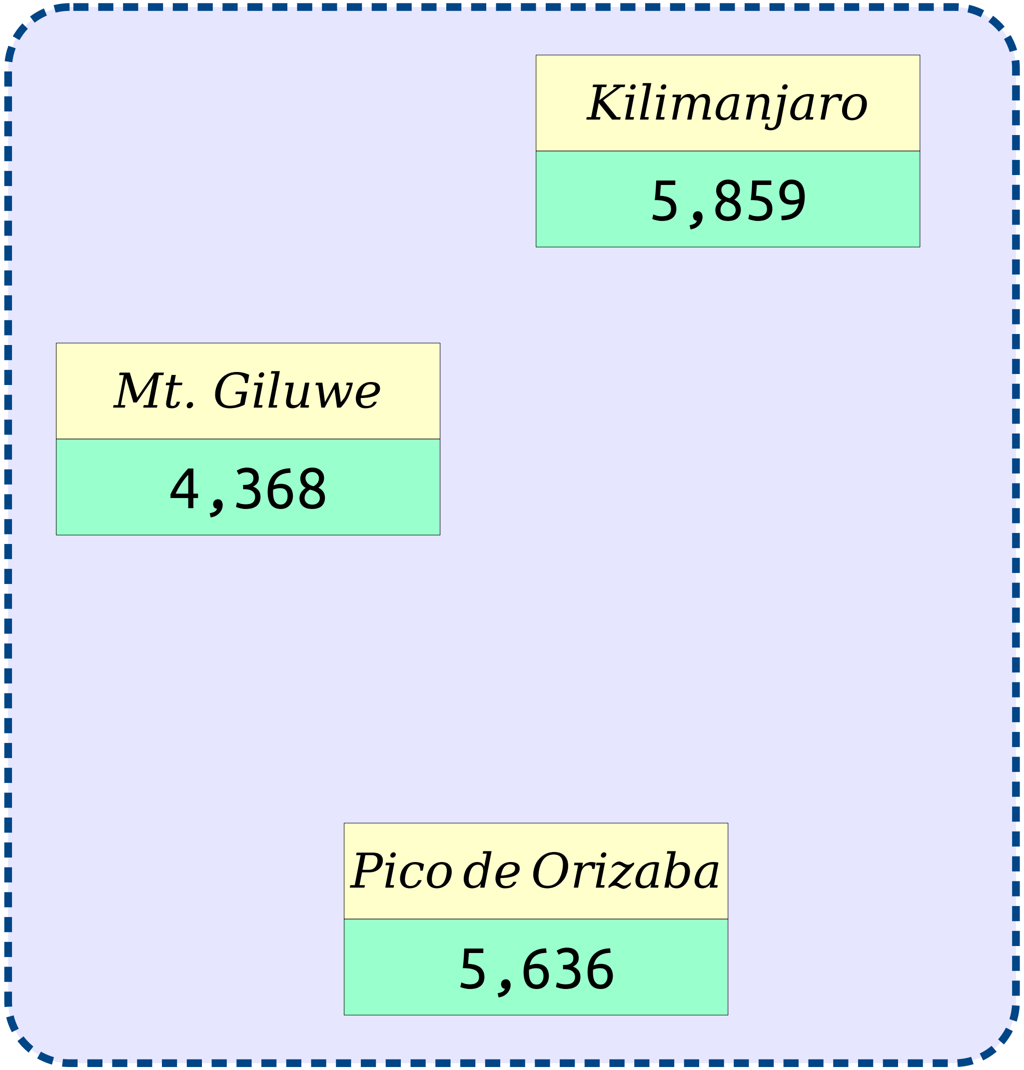
Priority Queues

- A ***priority queue*** is a data structure that supports these operations:
  - *pq*.***enqueue***(*v*, *k*), which enqueues element *v* with key *k*;
  - *pq*.***find-min***(), which returns the element with the least key; and
  - *pq*.***extract-min***(), which removes and returns the element with the least key.
- They're useful as building blocks in a *bunch* of algorithms.



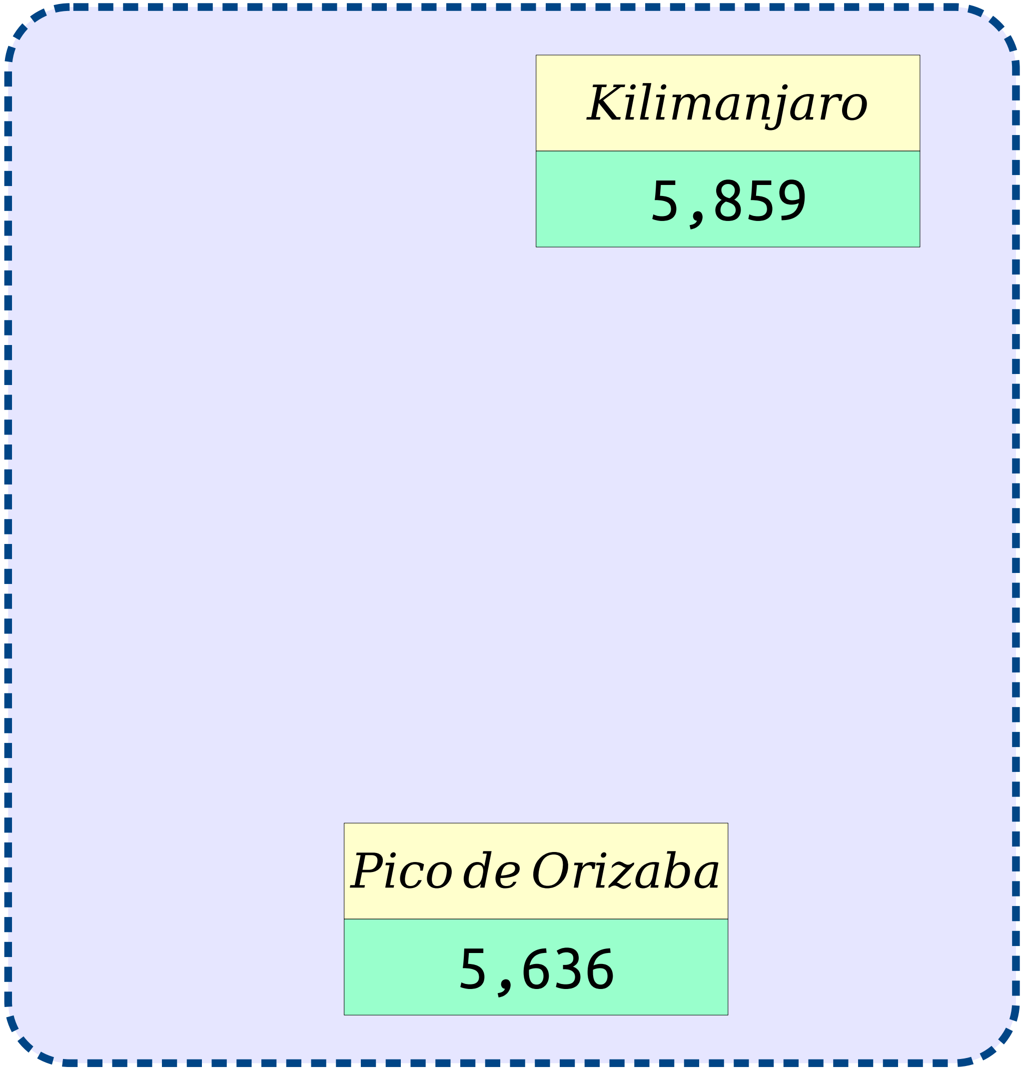
Priority Queues

- A ***priority queue*** is a data structure that supports these operations:
  - *pq*.***enqueue***(*v*, *k*), which enqueues element *v* with key *k*;
  - *pq*.***find-min***(), which returns the element with the least key; and
  - *pq*.***extract-min***(), which removes and returns the element with the least key.
- They're useful as building blocks in a *bunch* of algorithms.



Priority Queues

- A ***priority queue*** is a data structure that supports these operations:
  - *pq*.***enqueue***(*v*, *k*), which enqueues element *v* with key *k*;
  - *pq*.***find-min***(), which returns the element with the least key; and
  - *pq*.***extract-min***(), which removes and returns the element with the least key.
- They're useful as building blocks in a *bunch* of algorithms.



Binary Heaps

- Priority queues are often implemented as ***binary heaps***.
  - ***enqueue*** and ***extract-min*** run in time O(log *n*); ***find-min*** runs in time O(1).
- These heaps are surprisingly fast in practice. It's tough to beat their performance!
  - *d*-ary heaps outperform binary heaps for well-tuned values of *d*, and otherwise only ***sequence heaps*** are known to specifically outperform this family.
  - (Is this information incorrect as of 2025? Let me know and I'll update it.)
- In that case, why do we need other heaps?

Priority Queues in Practice

- Many graph algorithms directly rely on priority queues supporting extra operations:
  - ***meld***(pq_1, pq_2): Destroy pq_1 and pq_2 and combine their elements into a single priority queue. *(MSTs via Cheriton-Tarjan)*
  - *pq*.***decrease-key***(v, k'): Given a pointer to element v already in the queue, lower its key to have new value k'. *(Shortest paths via Dijkstra, global min-cut via Stoer-Wagner)*
  - *pq*.***add-to-all***(Δk): Add Δk to the keys of each element in the priority queue, typically used with ***meld***. *(Optimum branchings via Chu-Edmonds-Liu)*
- In lecture, we'll cover binomial heaps to efficiently support ***meld*** and Fibonacci heaps to efficiently support ***meld*** and ***decrease-key***.
- You'll design a priority queue supporting ***meld*** and ***add-to-all*** on the next problem set.

Priority Queues in Practice

Many graph algorithms directly rely on priority queues supporting extra operations:

- ***meld***(pq_1, pq_2): Destroy pq_1 and pq_2 and combine their elements into a single priority queue. *(MSTs via Cheriton-Tarjan)*

  pq.***decrease-key***(*v*, *k*'): Given a pointer to element *v* already in the queue, lower its key to have new value *k*'. *(Shortest paths via Dijkstra, global min-cut via Stoer-Wagner)*

  pq.***add-to-all***(Δ*k*): Add Δ*k* to the keys of each element in the priority queue, typically used with ***meld***. *(Optimum branchings via Chu-Edmonds-Liu)*

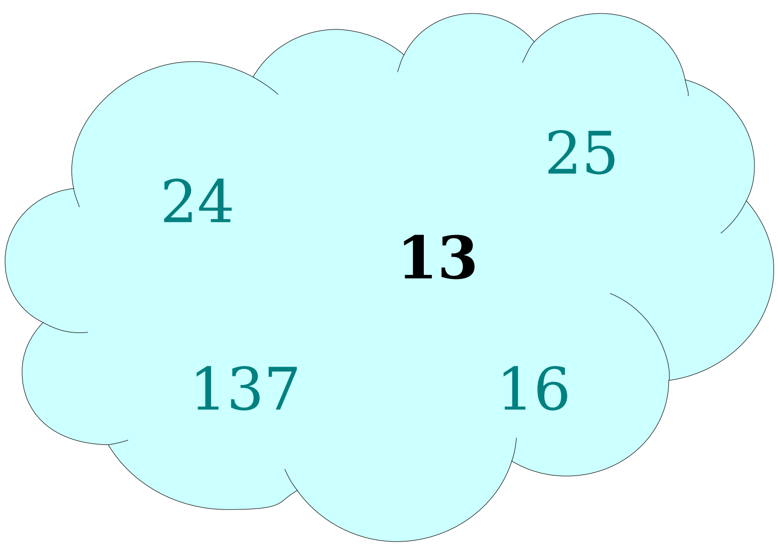
In lecture, we'll cover binomial heaps to efficiently support ***meld*** and Fibonacci heaps to efficiently support ***meld*** and ***decrease-key***.

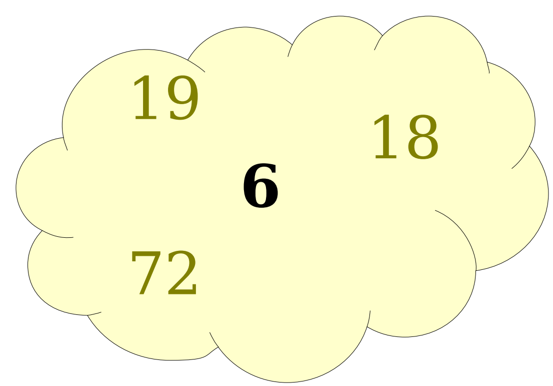
You’ll design a priority queue supporting ***meld*** and ***add-to-all*** on the next problem set.

Meldable Priority Queues

Meldable Priority Queues

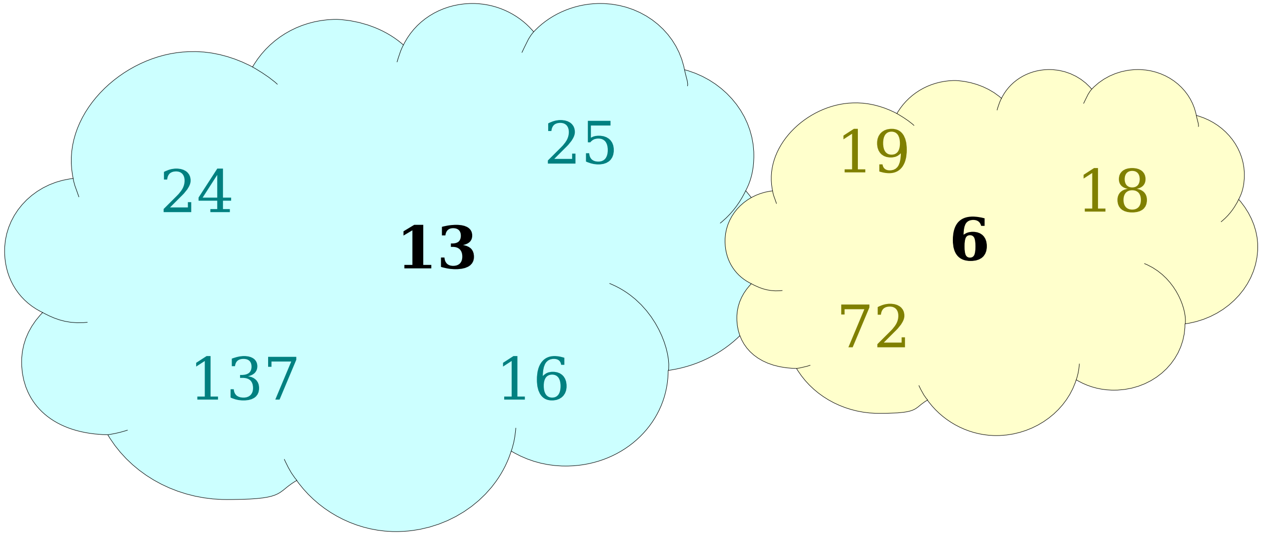
- A priority queue supporting the ***meld*** operation is called a ***meldable priority queue***.
- ***meld***(pq_1, pq_2) destructively modifies pq_1 and pq_2 and produces a new priority queue containing all elements of pq_1 and pq_2.





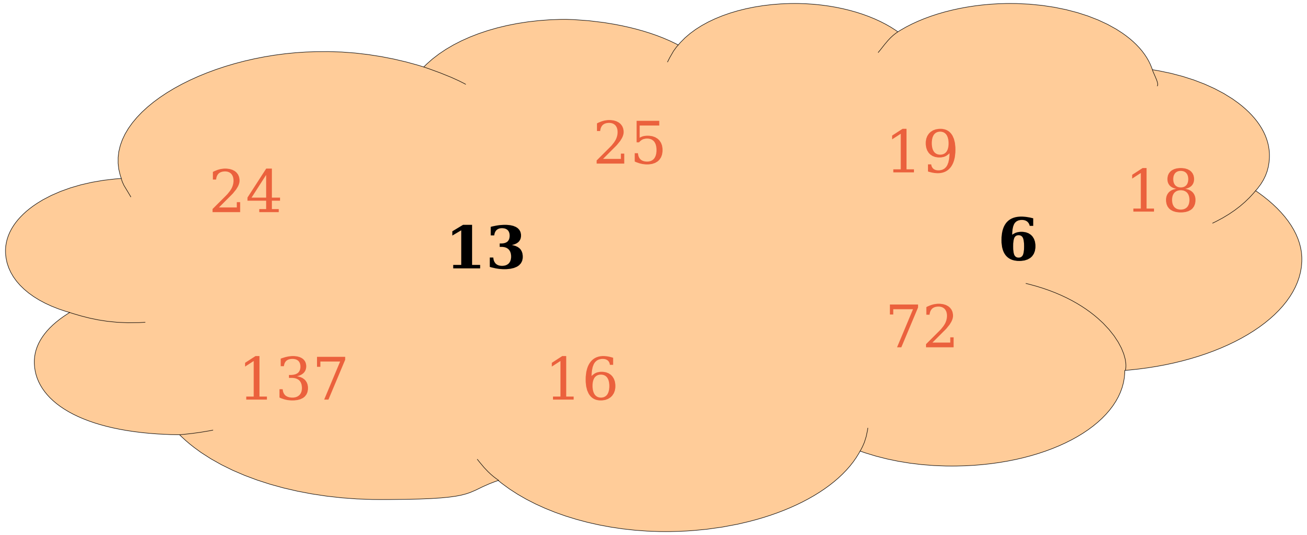
Meldable Priority Queues

- A priority queue supporting the ***meld*** operation is called a ***meldable priority queue***.
- ***meld***(pq_1, pq_2) destructively modifies pq_1 and pq_2 and produces a new priority queue containing all elements of pq_1 and pq_2.



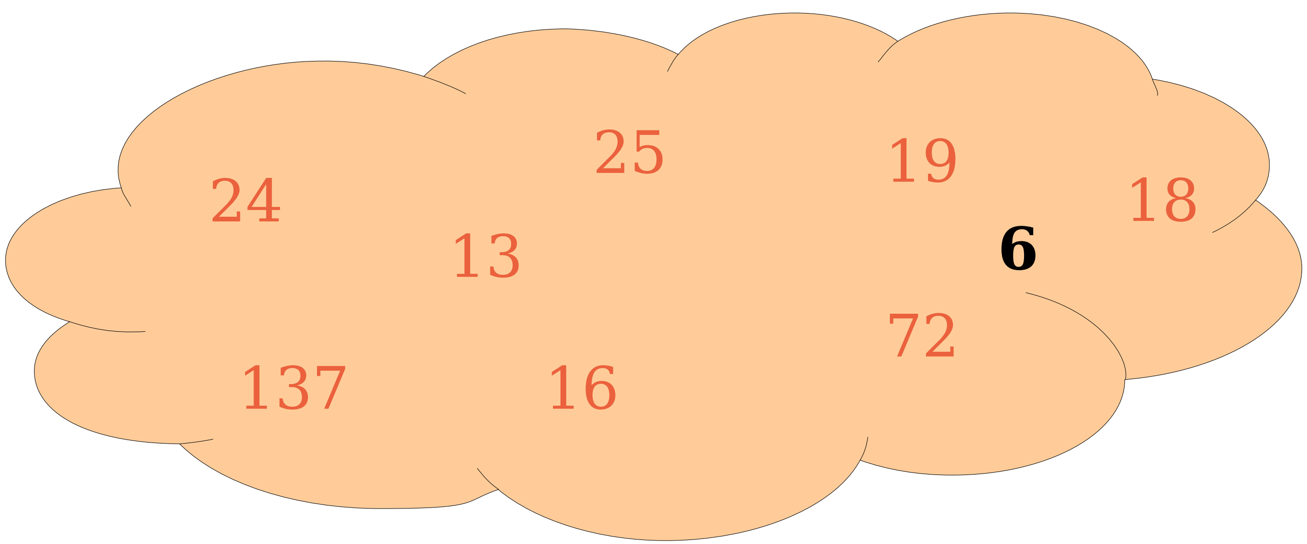
Meldable Priority Queues

- A priority queue supporting the ***meld*** operation is called a ***meldable priority queue***.
- ***meld***(pq_1, pq_2) destructively modifies pq_1 and pq_2 and produces a new priority queue containing all elements of pq_1 and pq_2.



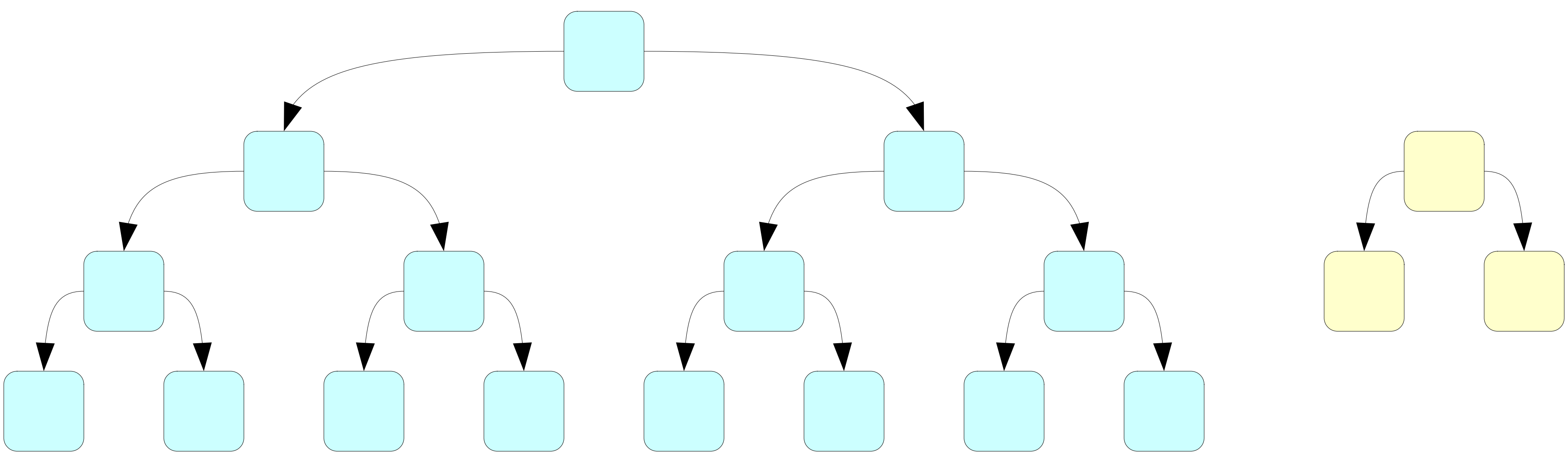
Meldable Priority Queues

- A priority queue supporting the ***meld*** operation is called a ***meldable priority queue***.
- ***meld***(pq_1, pq_2) destructively modifies pq_1 and pq_2 and produces a new priority queue containing all elements of pq_1 and pq_2.



Efficiently Meldable Queues

- Standard binary heaps do not efficiently support ***meld***.
- ***Intuition***: Binary heaps are complete binary trees, and two complete binary trees cannot easily be linked to one another.



What things *can* be combined together efficiently?

Adding Binary Numbers

- Given the binary representations of two numbers n and m, we can add those numbers in time $O(\log m + \log n)$.

Intuition:
Writing out n in any "reasonable" base requires $\Theta(\log n)$ digits.

Adding Binary Numbers

- Given the binary representations of two numbers *n* and *m*, we can add those numbers in time O(log *m* + log *n*).

	1	0	1	1	0
+		1	1	1	1

Adding Binary Numbers

- Given the binary representations of two numbers *n* and *m*, we can add those numbers in time O(log *m* + log *n*).

	1	0	1	1	0
+		1	1	1	1

Adding Binary Numbers

- Given the binary representations of two numbers *n* and *m*, we can add those numbers in time O(log *m* + log *n*).

	1	0	1	1	0
+		1	1	1	1
					1

Adding Binary Numbers

- Given the binary representations of two numbers n and m, we can add those numbers in time $O(\log m + \log n)$.

	1	0	1	1	0
+		1	1	1	1
					1

Adding Binary Numbers

- Given the binary representations of two numbers *n* and *m*, we can add those numbers in time O(log *m* + log *n*).

			1		
	1	0	1	1	0
+		1	1	1	1
				0	1

Adding Binary Numbers

- Given the binary representations of two numbers *n* and *m*, we can add those numbers in time O(log *m* + log *n*).

			1		
	1	0	1	1	0
+		1	1	1	1
				0	1

Adding Binary Numbers

- Given the binary representations of two numbers *n* and *m*, we can add those numbers in time O(log *m* + log *n*).

		1	1		
	1	0	1	1	0
+		1	1	1	1
			1	0	1

Adding Binary Numbers

- Given the binary representations of two numbers *n* and *m*, we can add those numbers in time O(log *m* + log *n*).

		1	1		
	1	0	1	1	0
+		1	1	1	1
			1	0	1

Adding Binary Numbers

- Given the binary representations of two numbers *n* and *m*, we can add those numbers in time O(log *m* + log *n*).

	1	1	1		
	1	0	1	1	0
+		1	1	1	1
		0	1	0	1

Adding Binary Numbers

- Given the binary representations of two numbers *n* and *m*, we can add those numbers in time O(log *m* + log *n*).

	1	1	1		
	1	0	1	1	0
+		1	1	1	1
		0	1	0	1

Adding Binary Numbers

- Given the binary representations of two numbers *n* and *m*, we can add those numbers in time O(log *m* + log *n*).

1	1	1	1		
	1	0	1	1	0
+		1	1	1	1
	0	0	1	0	1

Adding Binary Numbers

- Given the binary representations of two numbers *n* and *m*, we can add those numbers in time O(log *m* + log *n*).

1	1	1	1		
	1	0	1	1	0
+		1	1	1	1
1	0	0	1	0	1

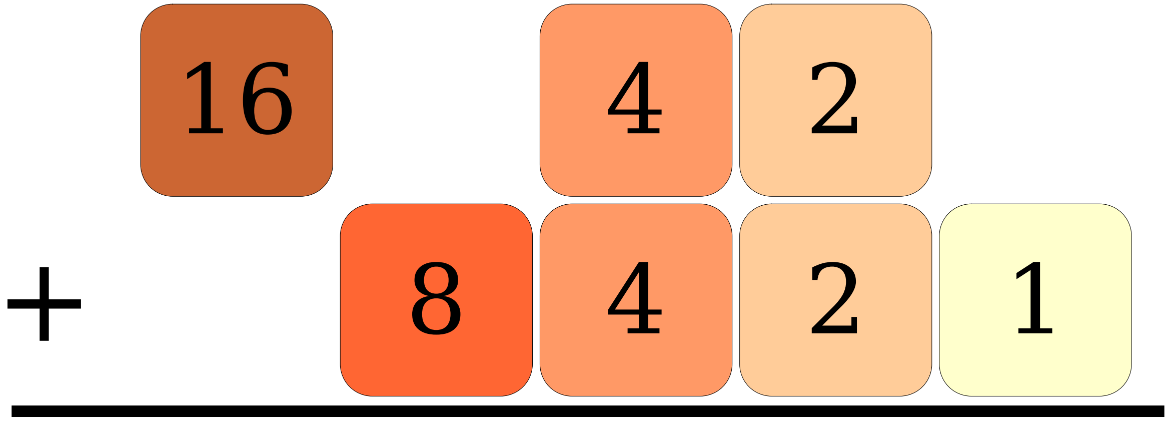
A Different Intuition

- Represent *n* and *m* as a collection of “packets” whose sizes are powers of two.
- Adding together *n* and *m* can then be thought of as combining the packets together, eliminating duplicates

$$\begin{array}{r} 1\ 0\ 1\ 1\ 0 \\ +\quad 1\ 1\ 1\ 1 \\ \hline \end{array}$$

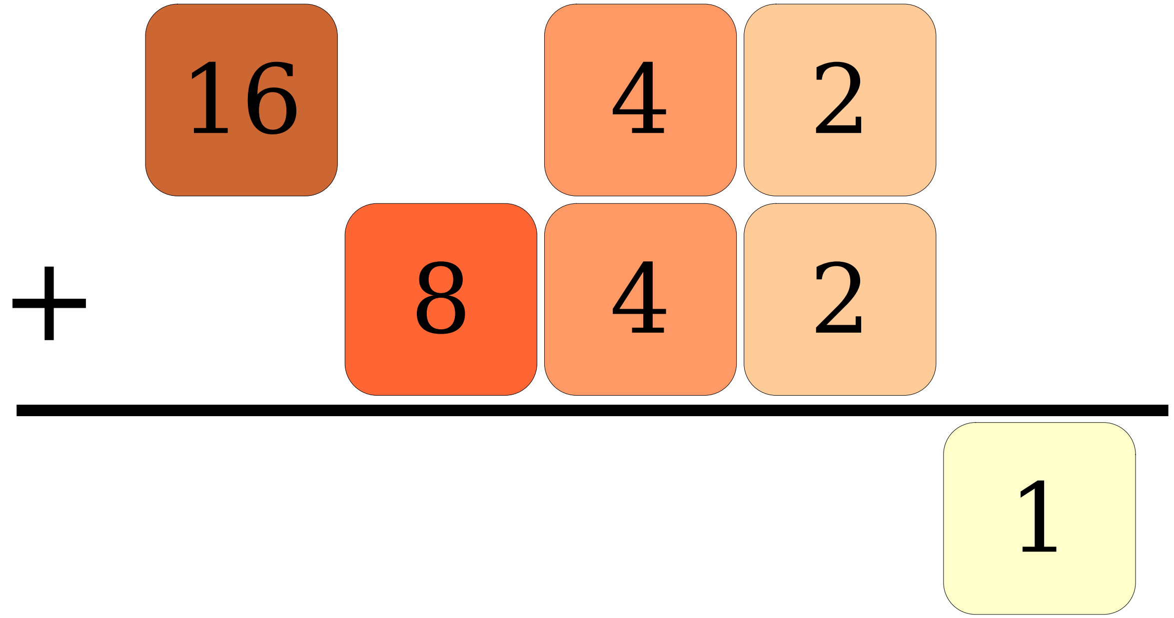
A Different Intuition

- Represent n and m as a collection of “packets” whose sizes are powers of two.
- Adding together n and m can then be thought of as combining the packets together, eliminating duplicates



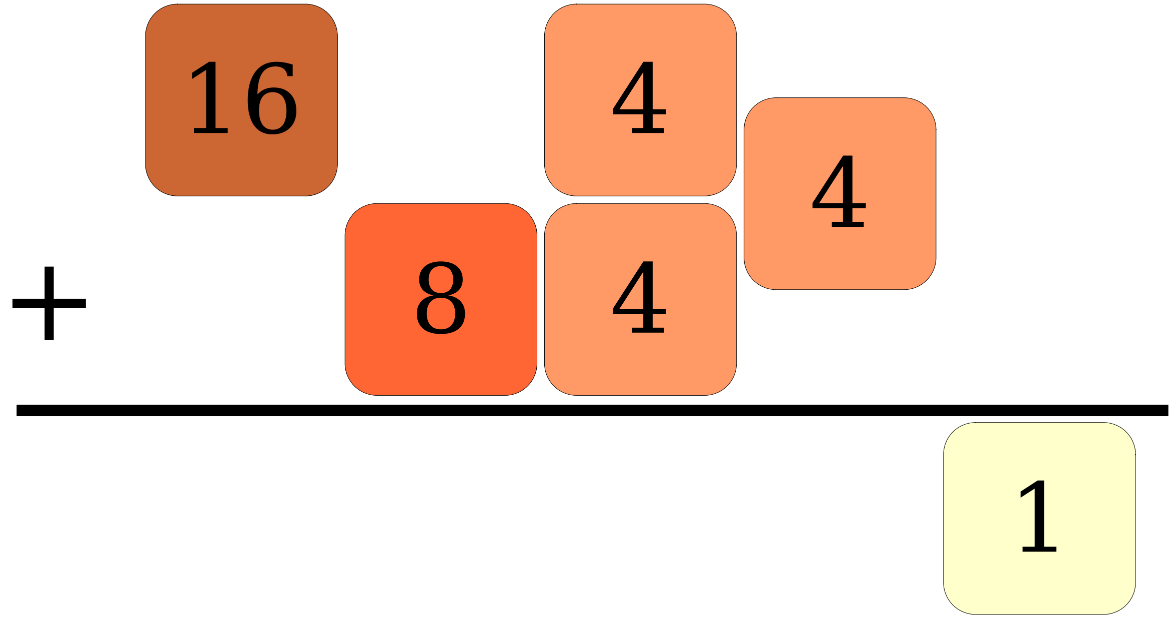
A Different Intuition

- Represent *n* and *m* as a collection of “packets” whose sizes are powers of two.
- Adding together *n* and *m* can then be thought of as combining the packets together, eliminating duplicates



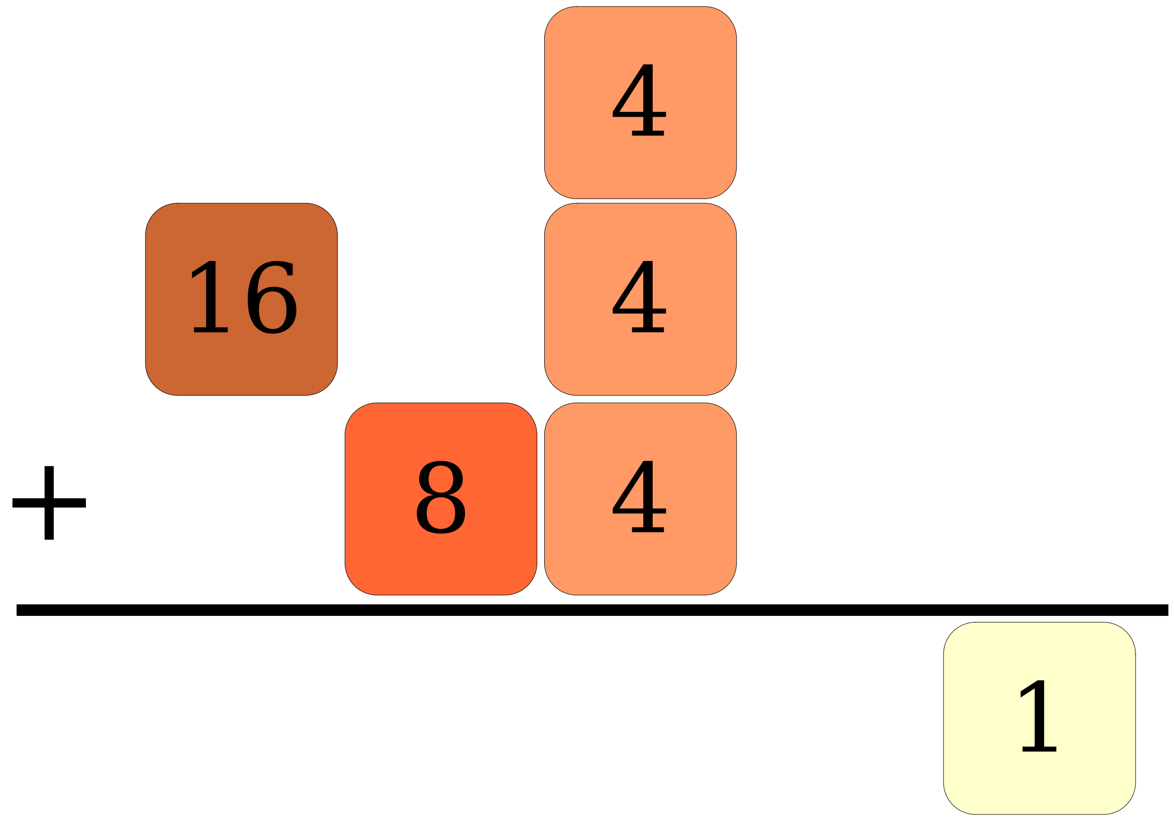
A Different Intuition

- Represent *n* and *m* as a collection of “packets” whose sizes are powers of two.
- Adding together *n* and *m* can then be thought of as combining the packets together, eliminating duplicates



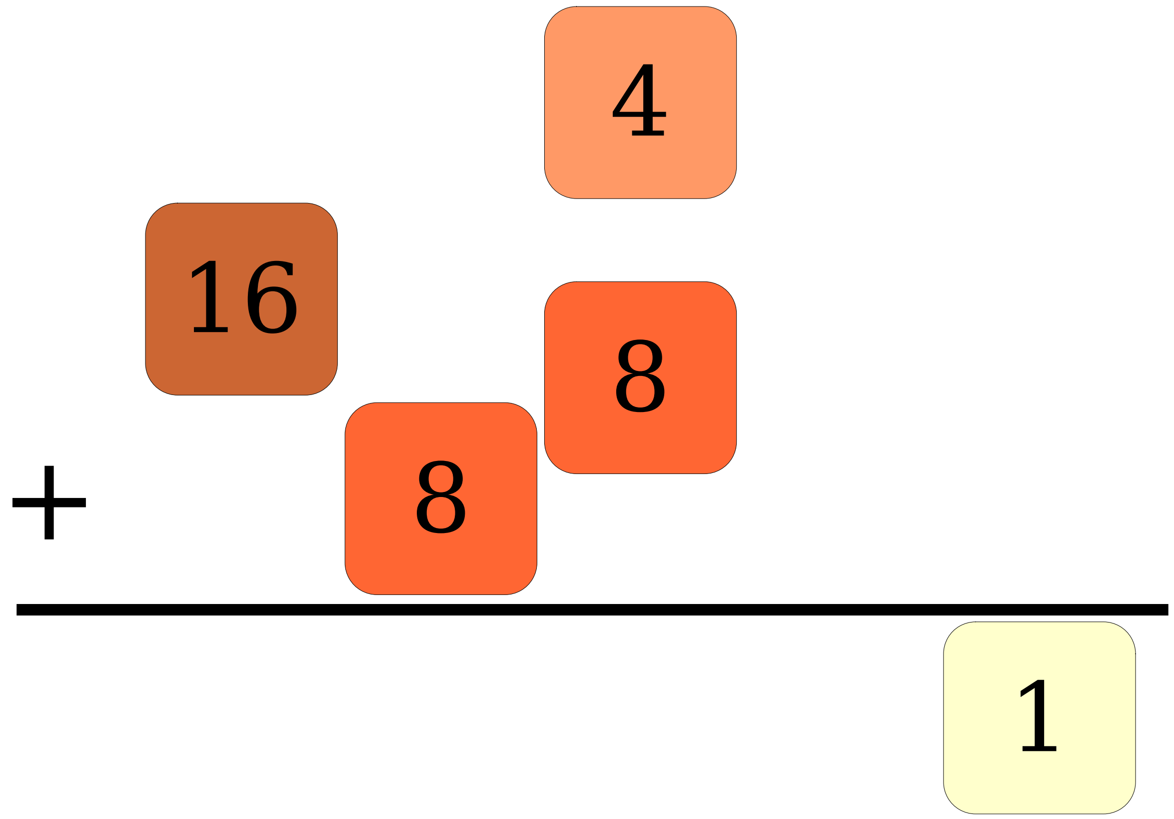
A Different Intuition

- Represent *n* and *m* as a collection of “packets” whose sizes are powers of two.
- Adding together *n* and *m* can then be thought of as combining the packets together, eliminating duplicates



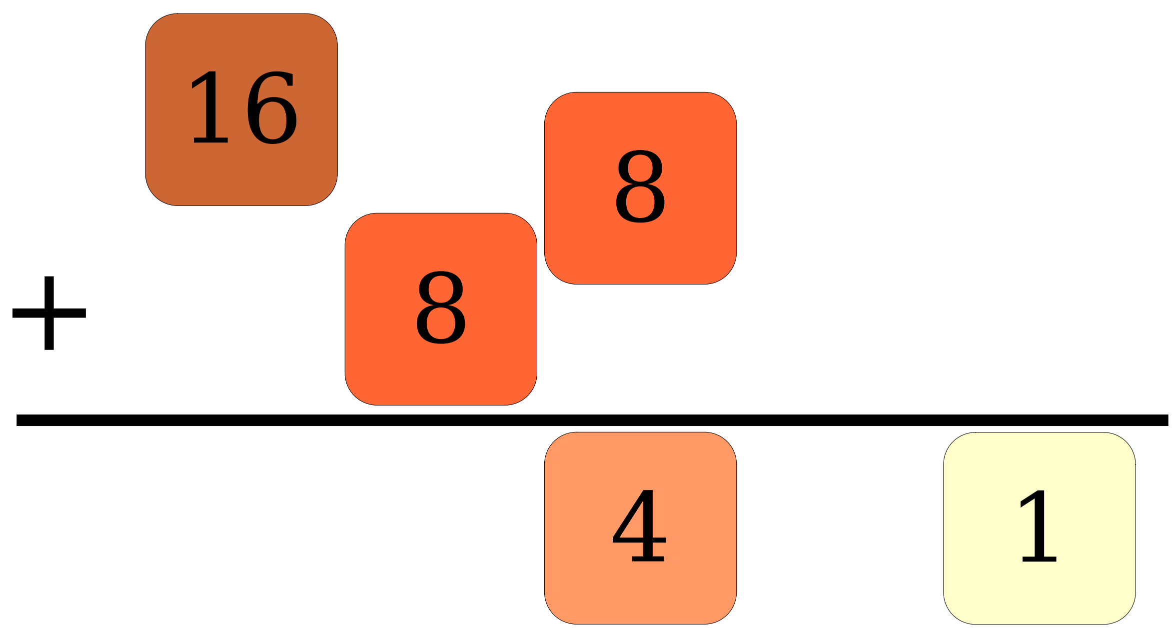
A Different Intuition

- Represent *n* and *m* as a collection of “packets” whose sizes are powers of two.
- Adding together *n* and *m* can then be thought of as combining the packets together, eliminating duplicates



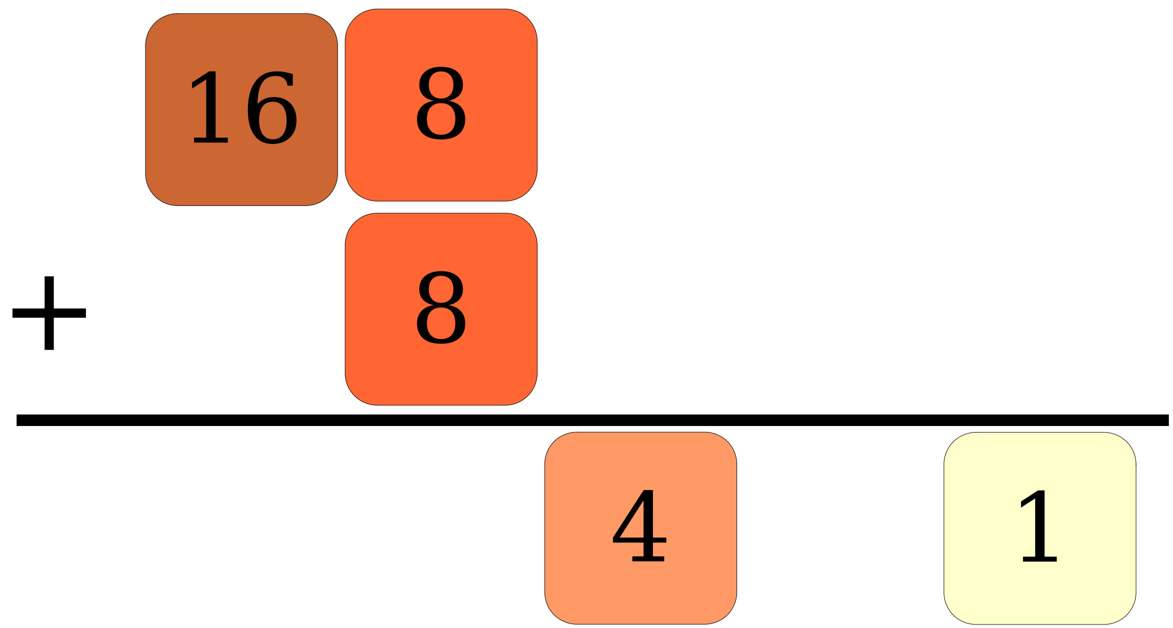
A Different Intuition

- Represent *n* and *m* as a collection of “packets” whose sizes are powers of two.
- Adding together *n* and *m* can then be thought of as combining the packets together, eliminating duplicates



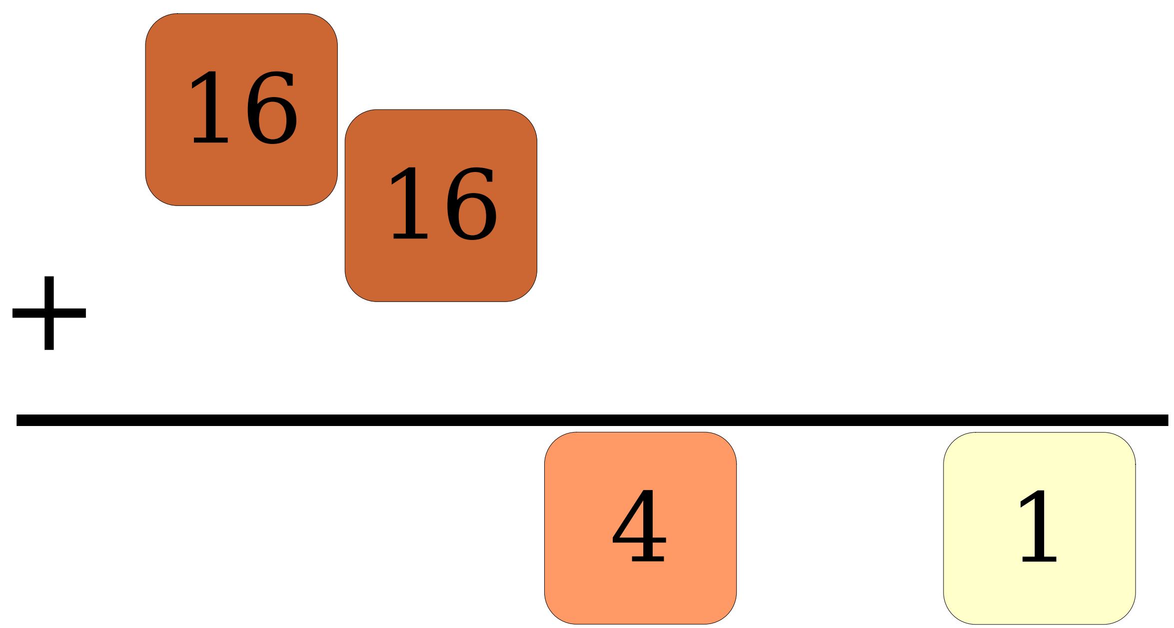
A Different Intuition

- Represent *n* and *m* as a collection of “packets” whose sizes are powers of two.
- Adding together *n* and *m* can then be thought of as combining the packets together, eliminating duplicates



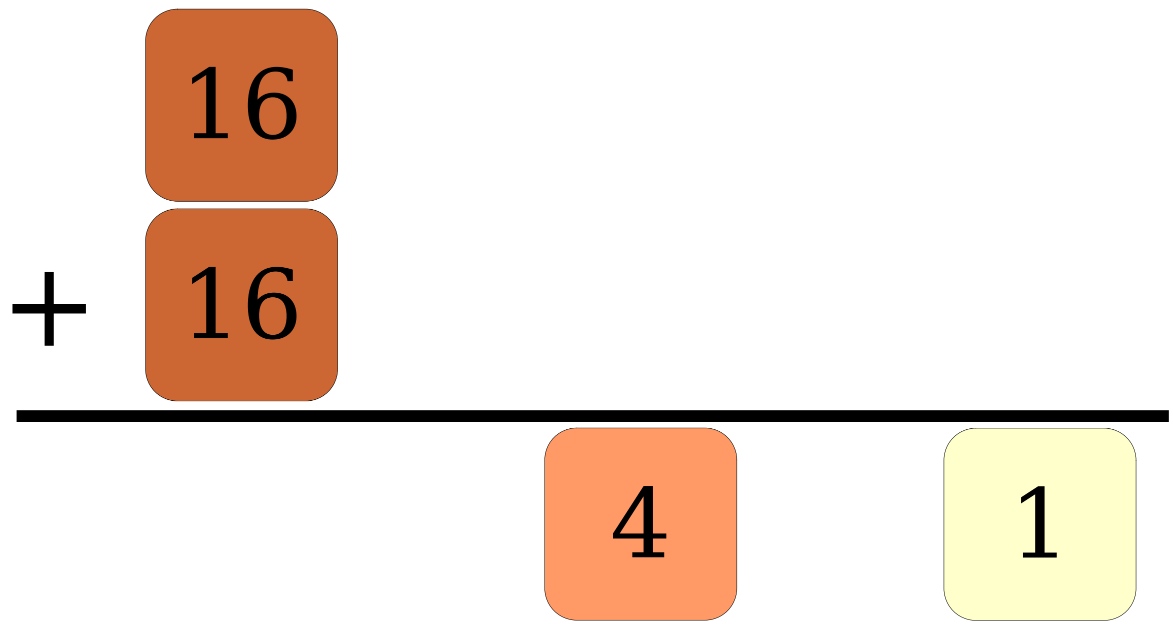
A Different Intuition

- Represent *n* and *m* as a collection of “packets” whose sizes are powers of two.
- Adding together *n* and *m* can then be thought of as combining the packets together, eliminating duplicates



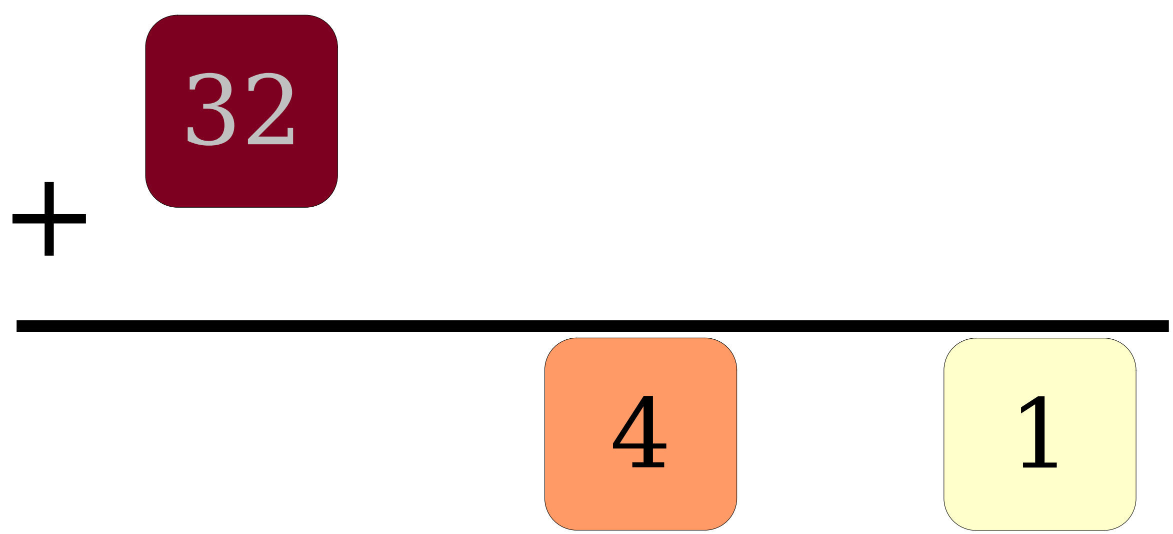
A Different Intuition

- Represent *n* and *m* as a collection of "packets" whose sizes are powers of two.
- Adding together *n* and *m* can then be thought of as combining the packets together, eliminating duplicates



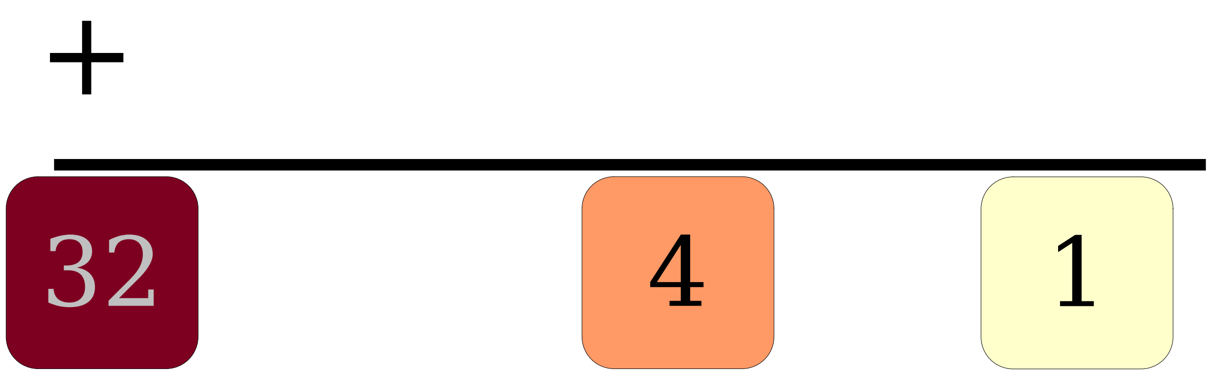
A Different Intuition

- Represent *n* and *m* as a collection of “packets” whose sizes are powers of two.
- Adding together *n* and *m* can then be thought of as combining the packets together, eliminating duplicates



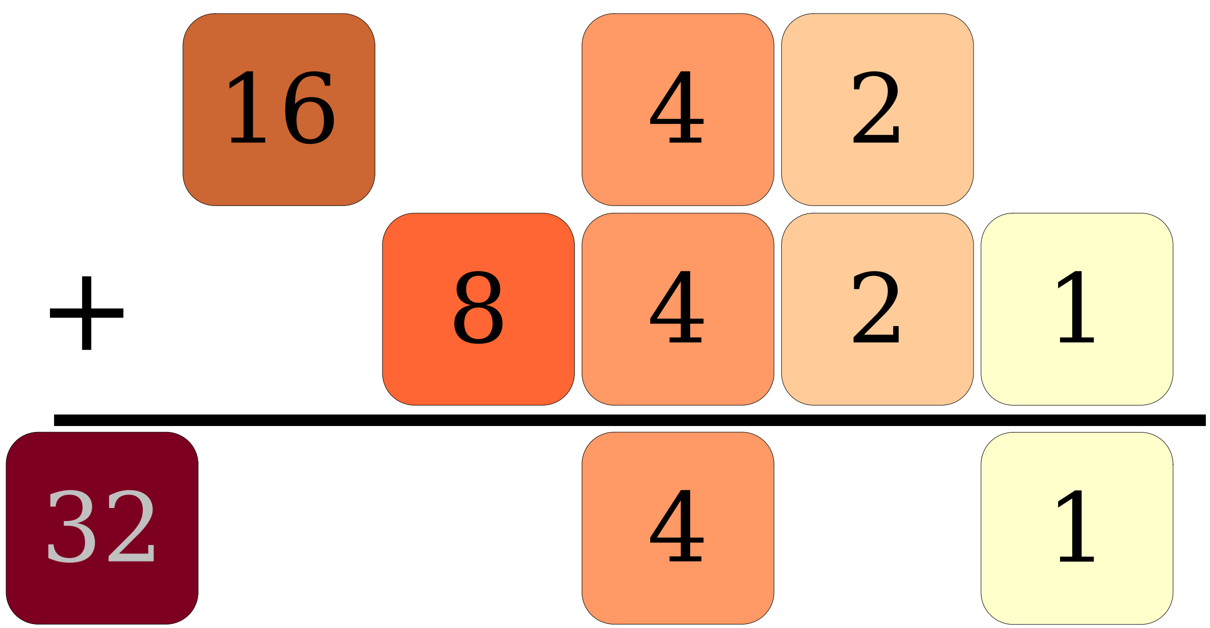
A Different Intuition

- Represent *n* and *m* as a collection of “packets” whose sizes are powers of two.
- Adding together *n* and *m* can then be thought of as combining the packets together, eliminating duplicates



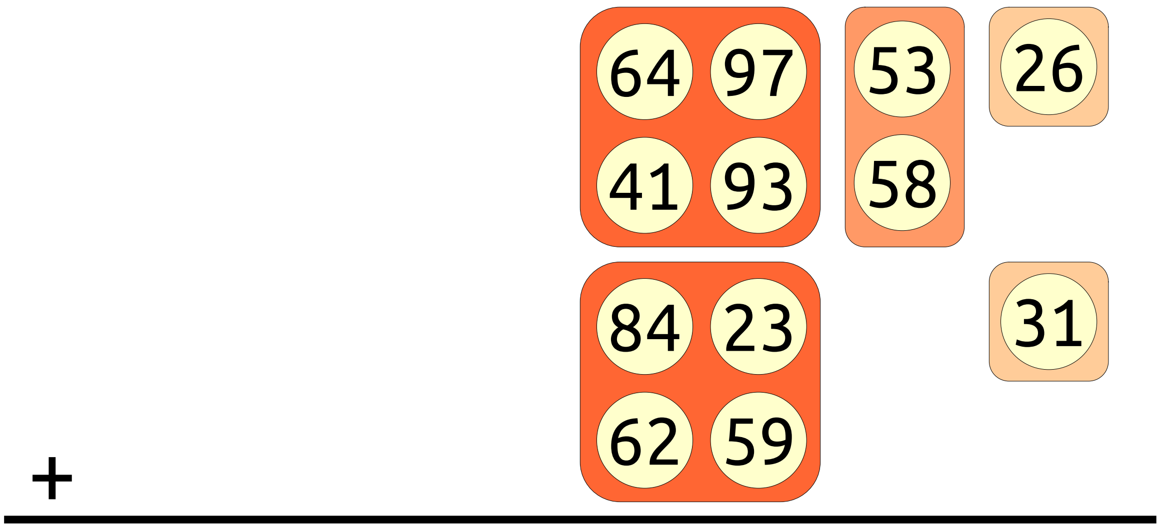
A Different Intuition

- Represent *n* and *m* as a collection of "packets" whose sizes are powers of two.
- Adding together *n* and *m* can then be thought of as combining the packets together, eliminating duplicates



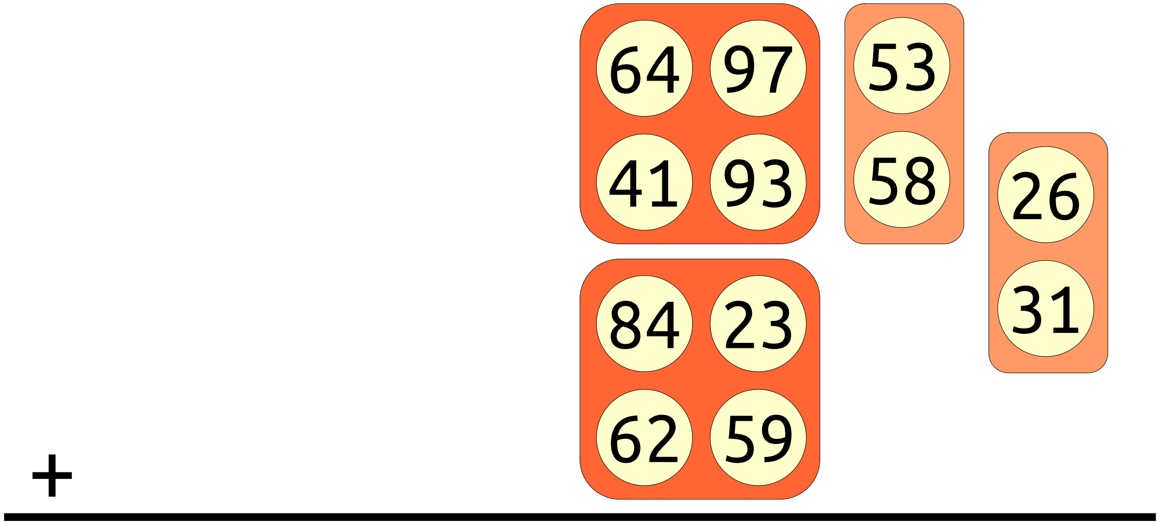
Building a Priority Queue

- ***Idea:*** Store elements in "packets" whose sizes are powers of two and ***meld*** by "adding" groups of packets.



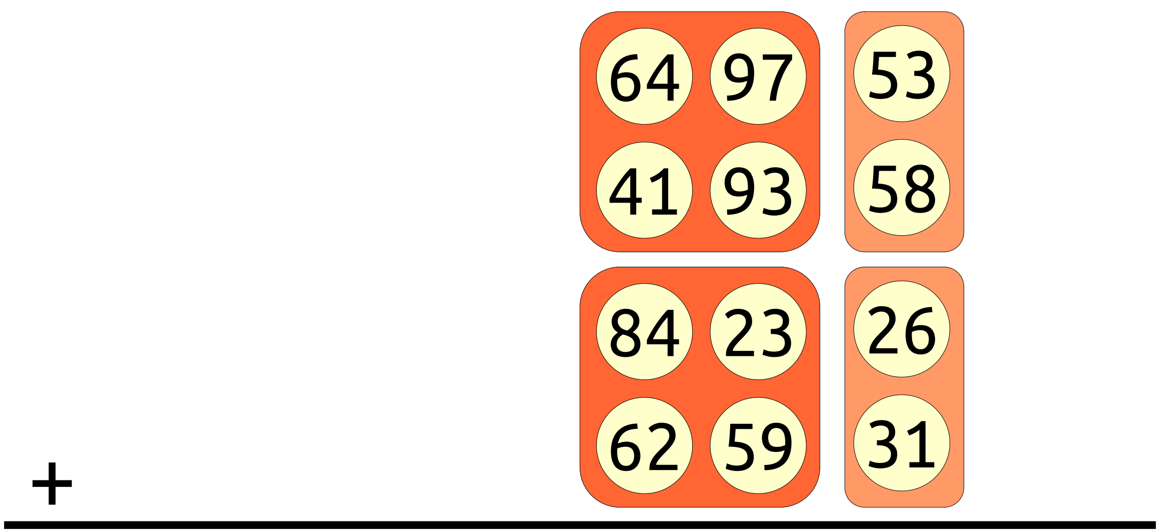
Building a Priority Queue

- ***Idea:*** Store elements in "packets" whose sizes are powers of two and ***meld*** by "adding" groups of packets.



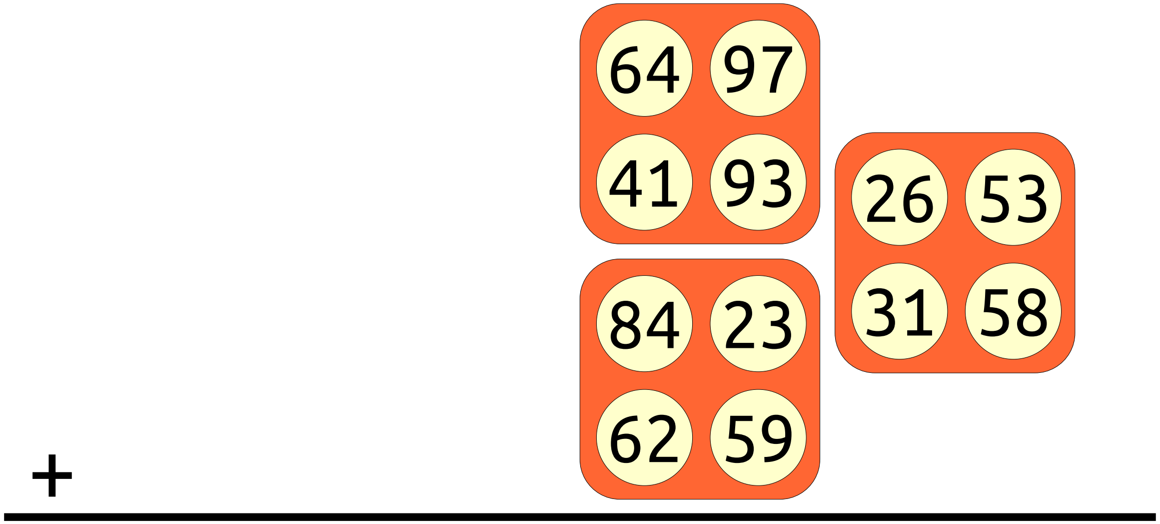
Building a Priority Queue

- ***Idea:*** Store elements in "packets" whose sizes are powers of two and ***meld*** by "adding" groups of packets.



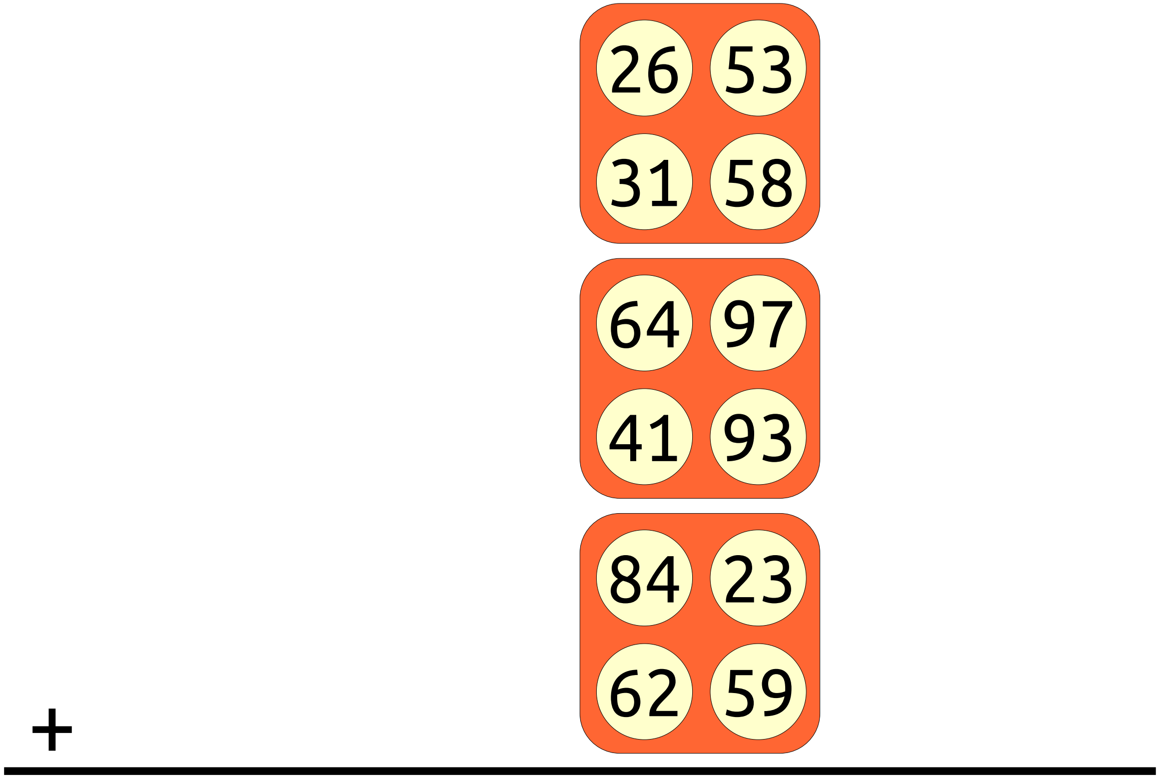
Building a Priority Queue

- ***Idea:*** Store elements in “packets” whose sizes are powers of two and ***meld*** by “adding” groups of packets.



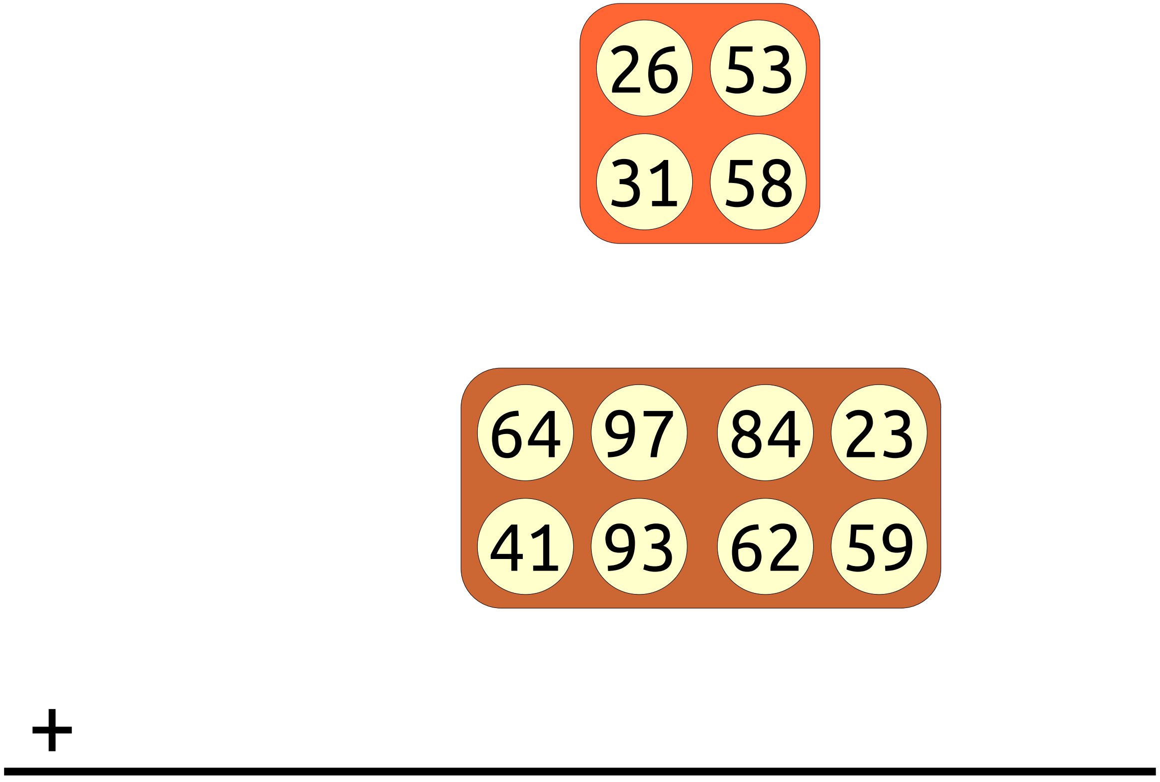
Building a Priority Queue

- ***Idea:*** Store elements in "packets" whose sizes are powers of two and ***meld*** by "adding" groups of packets.



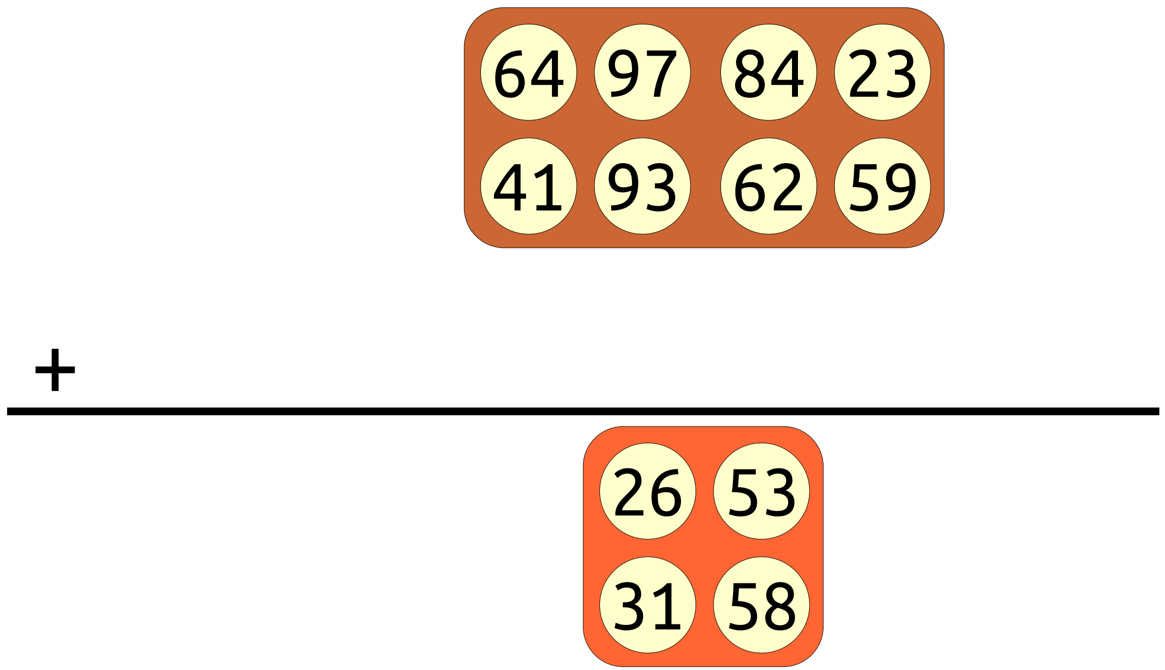
Building a Priority Queue

- ***Idea:*** Store elements in “packets” whose sizes are powers of two and ***meld*** by “adding” groups of packets.



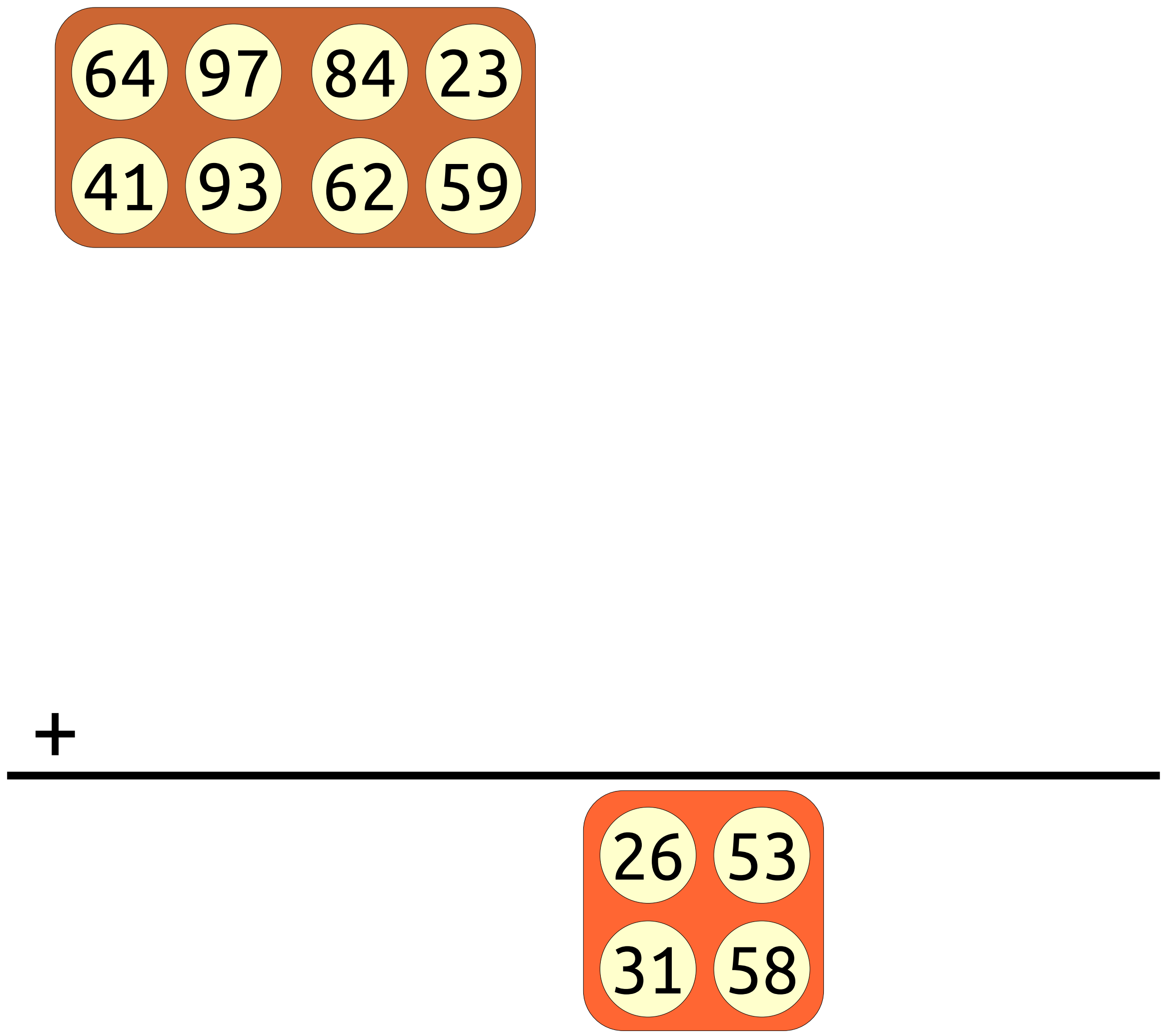
Building a Priority Queue

- ***Idea:*** Store elements in "packets" whose sizes are powers of two and ***meld*** by "adding" groups of packets.



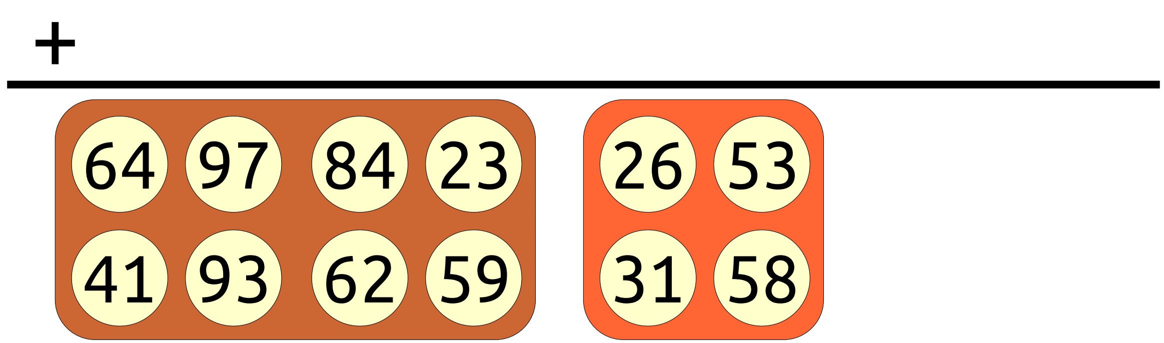
Building a Priority Queue

- ***Idea:*** Store elements in “packets” whose sizes are powers of two and ***meld*** by “adding” groups of packets.



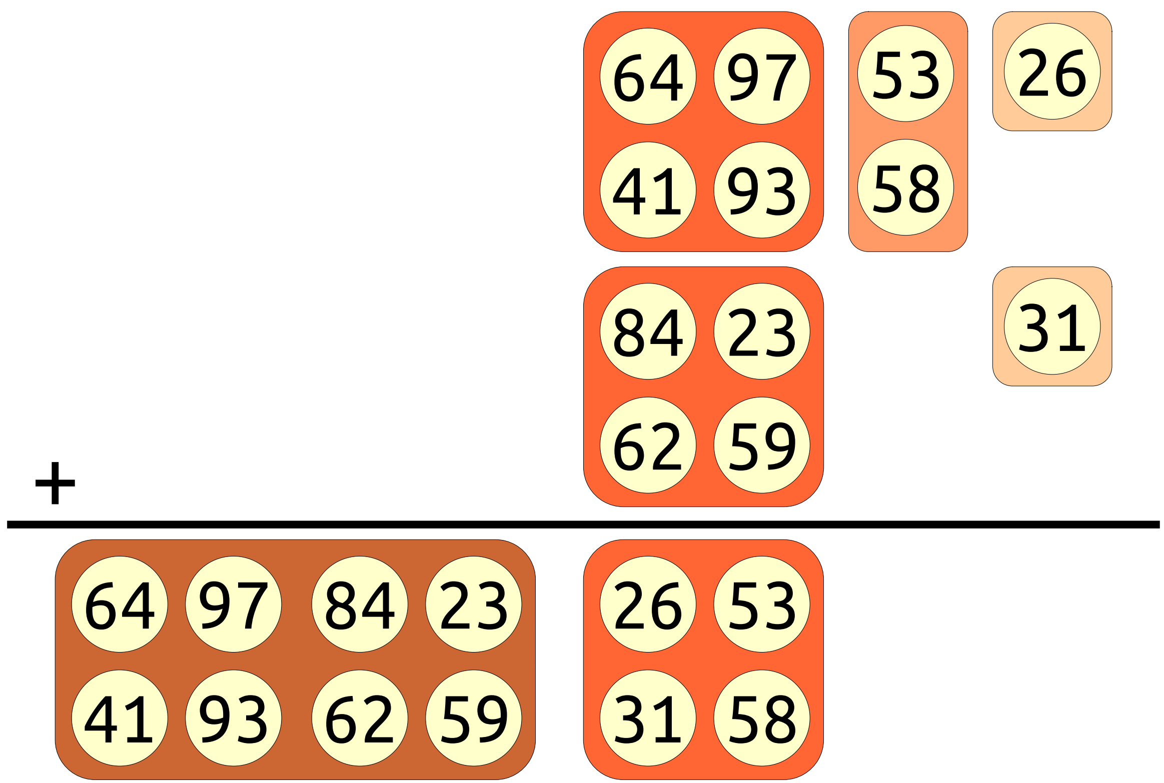
Building a Priority Queue

- ***Idea:*** Store elements in "packets" whose sizes are powers of two and ***meld*** by "adding" groups of packets.



Building a Priority Queue

- ***Idea:*** Store elements in "packets" whose sizes are powers of two and ***meld*** by "adding" groups of packets.

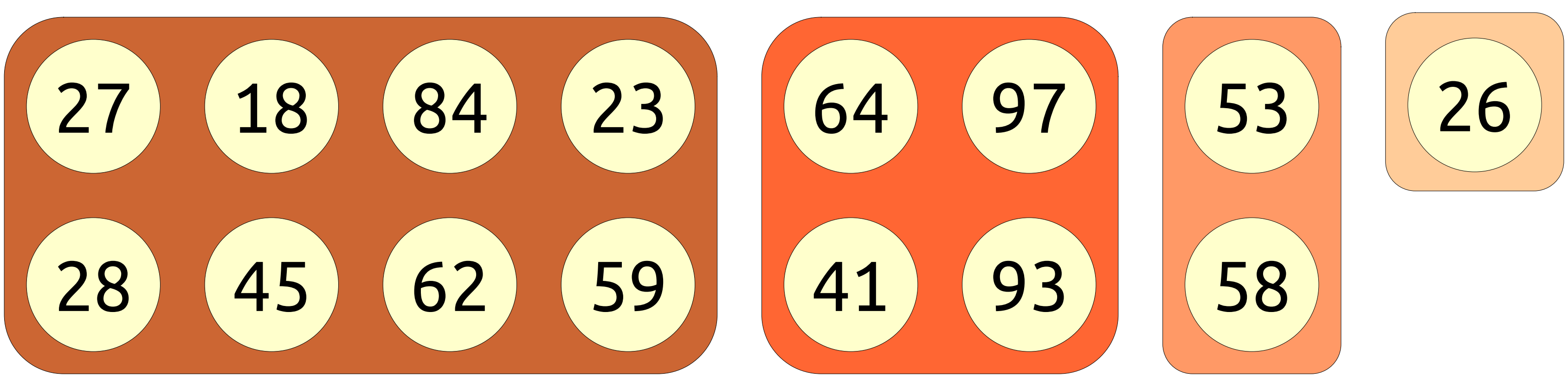


Building a Priority Queue

- What properties must our packets have?

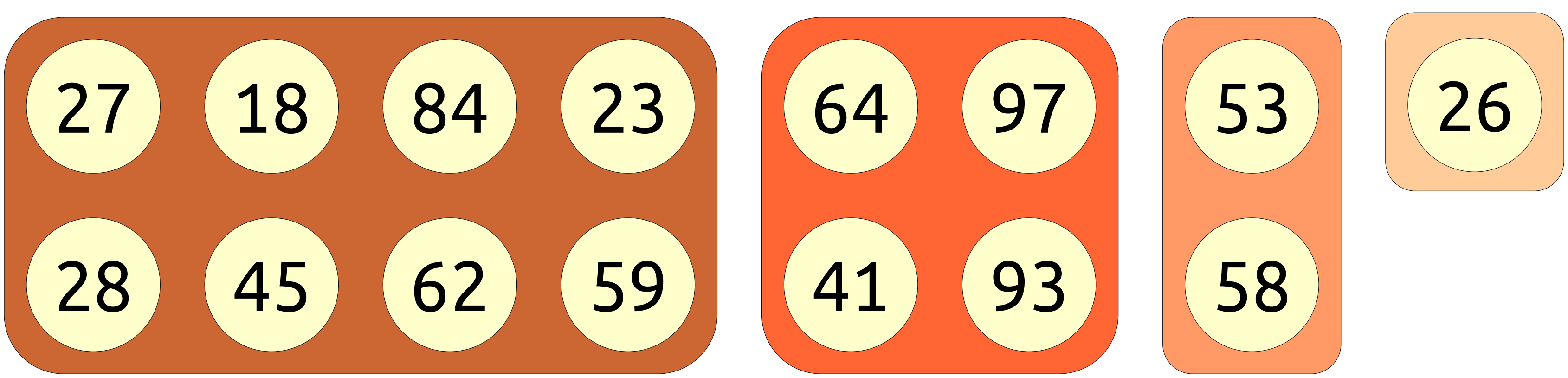
Building a Priority Queue

- What properties must our packets have?
  - Sizes must be powers of two.



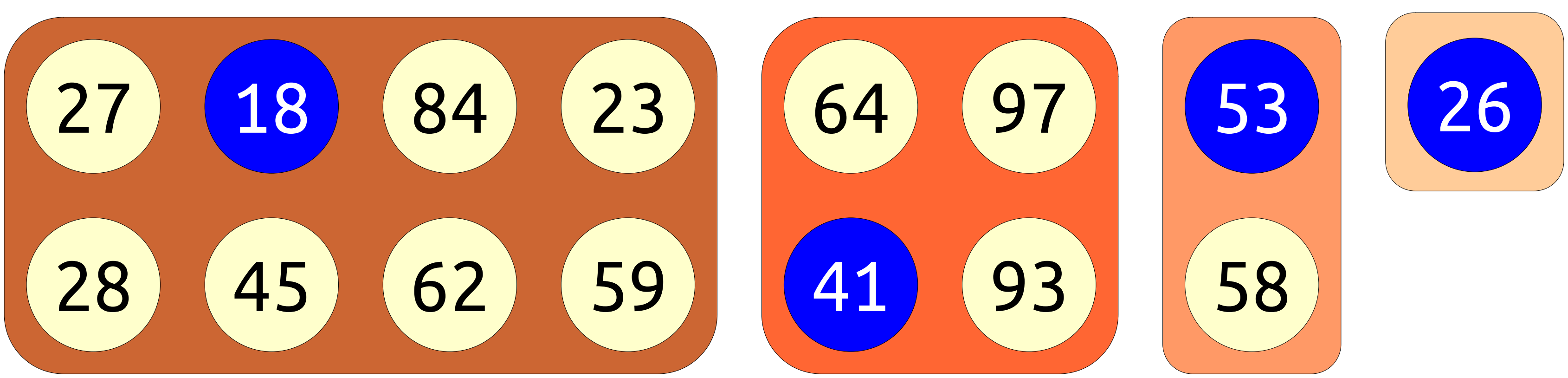
Building a Priority Queue

- What properties must our packets have?
  - Sizes must be powers of two.
  - Can efficiently fuse packets of the same size.



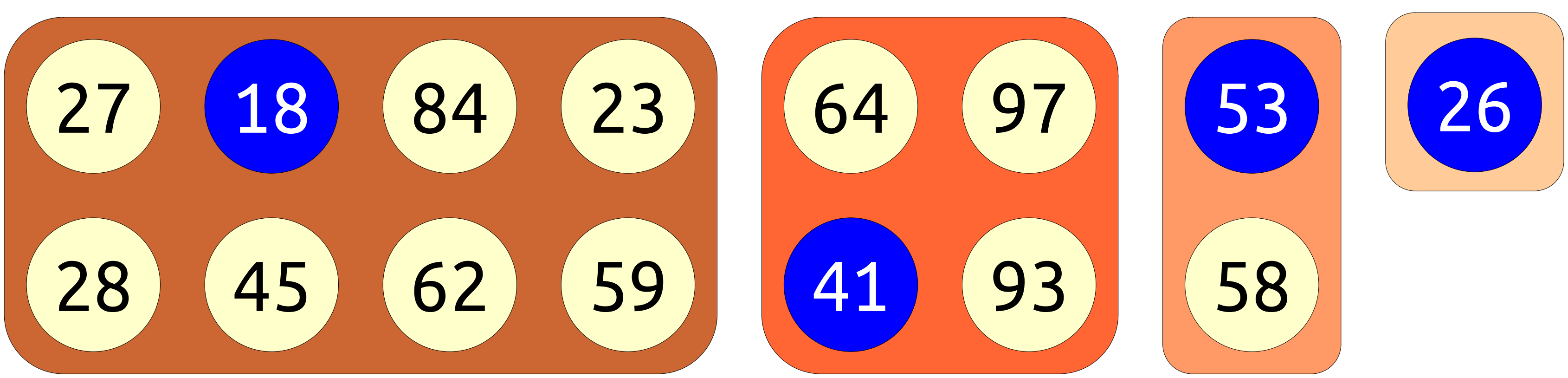
Building a Priority Queue

- What properties must our packets have?
  - Sizes must be powers of two.
  - Can efficiently fuse packets of the same size.



Building a Priority Queue

- What properties must our packets have?
  - Sizes must be powers of two.
  - Can efficiently fuse packets of the same size.
  - Can efficiently find the minimum element of each packet.

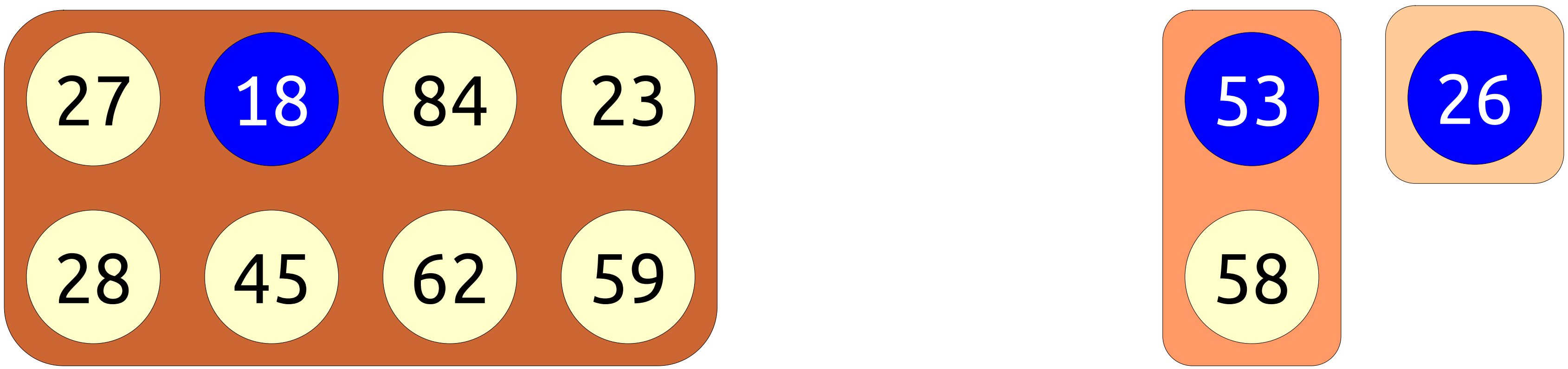


Inserting into the Queue

- If we can efficiently meld two priority queues, we can efficiently enqueue elements to the queue.
- ***Idea:*** Meld together the queue and a new queue with a single packet.

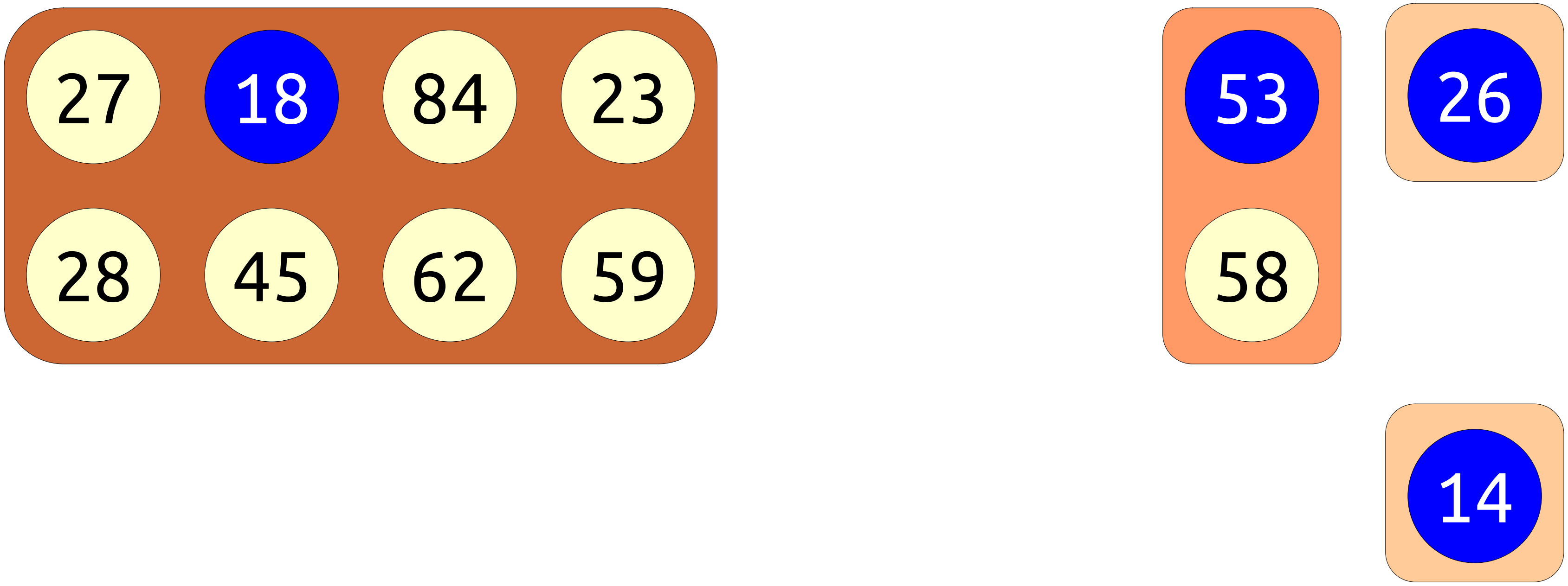
Inserting into the Queue

- If we can efficiently meld two priority queues, we can efficiently enqueue elements to the queue.
- ***Idea:*** Meld together the queue and a new queue with a single packet.



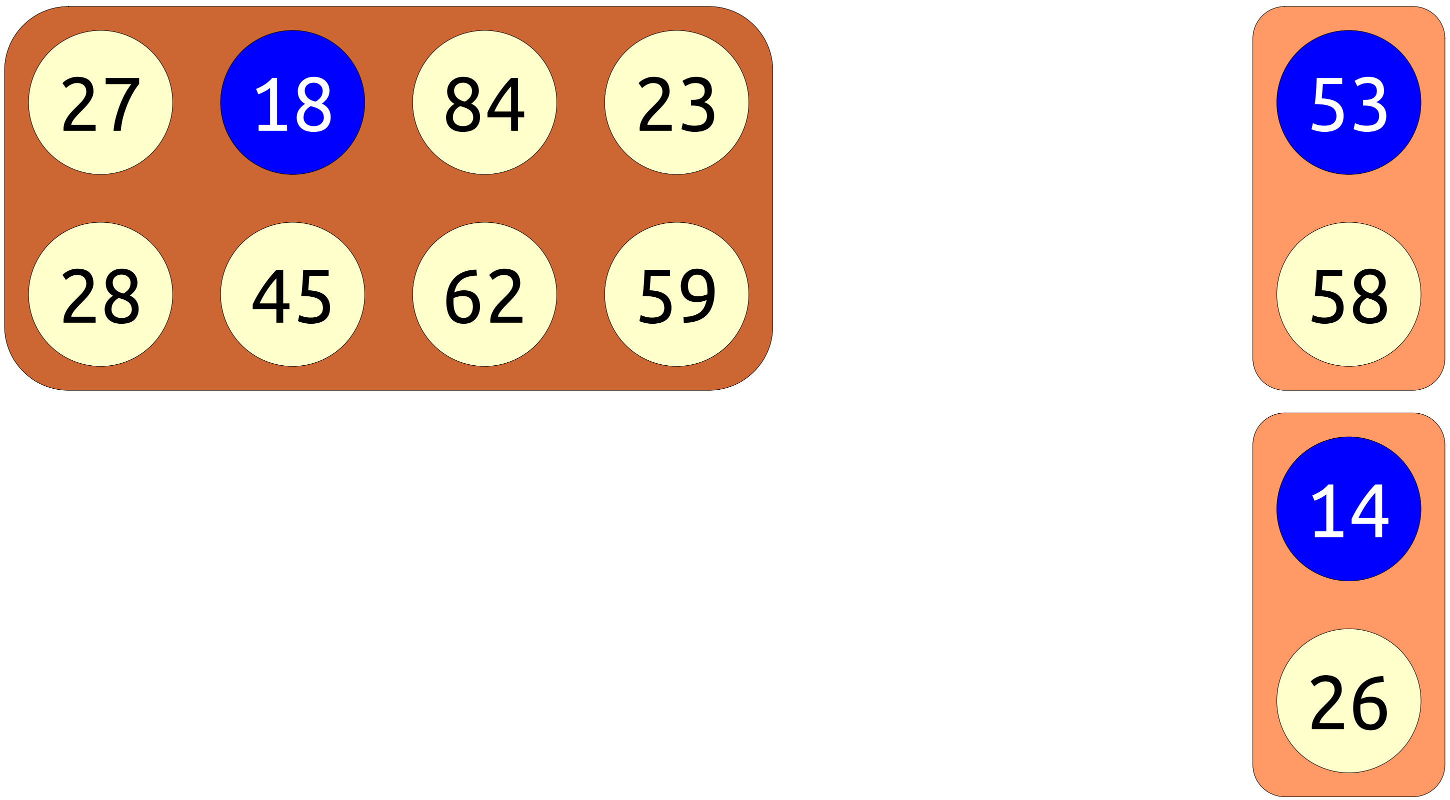
Inserting into the Queue

- If we can efficiently meld two priority queues, we can efficiently enqueue elements to the queue.
- ***Idea:*** Meld together the queue and a new queue with a single packet.



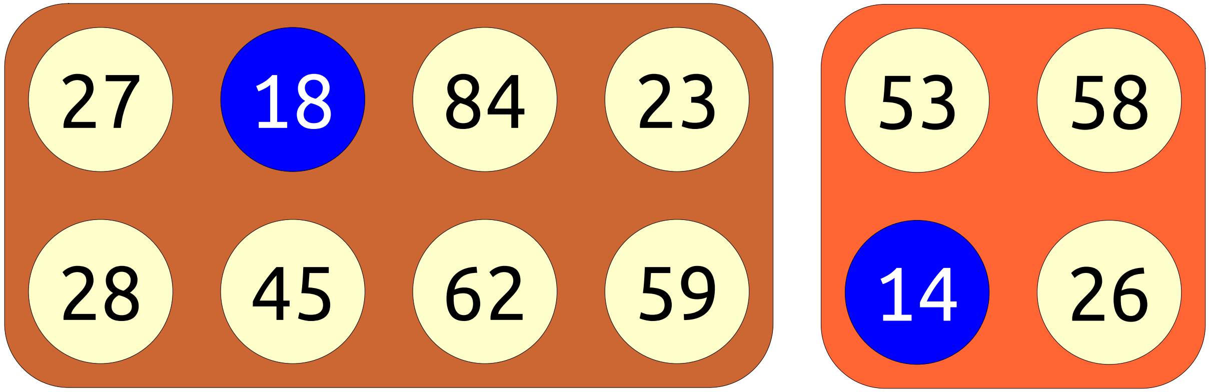
Inserting into the Queue

- If we can efficiently meld two priority queues, we can efficiently enqueue elements to the queue.
- ***Idea:*** Meld together the queue and a new queue with a single packet.



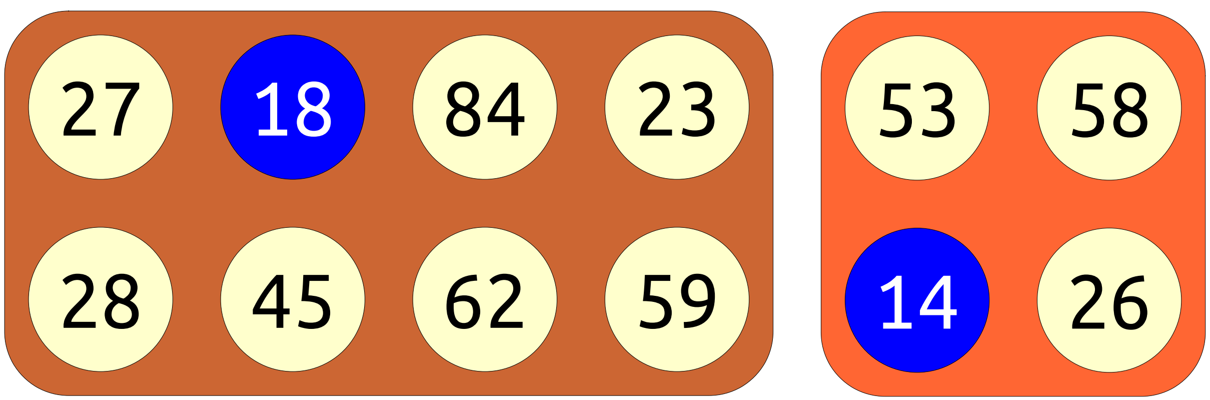
Inserting into the Queue

- If we can efficiently meld two priority queues, we can efficiently enqueue elements to the queue.
- ***Idea:*** Meld together the queue and a new queue with a single packet.



Inserting into the Queue

- If we can efficiently meld two priority queues, we can efficiently enqueue elements to the queue.
- ***Idea:*** Meld together the queue and a new queue with a single packet.



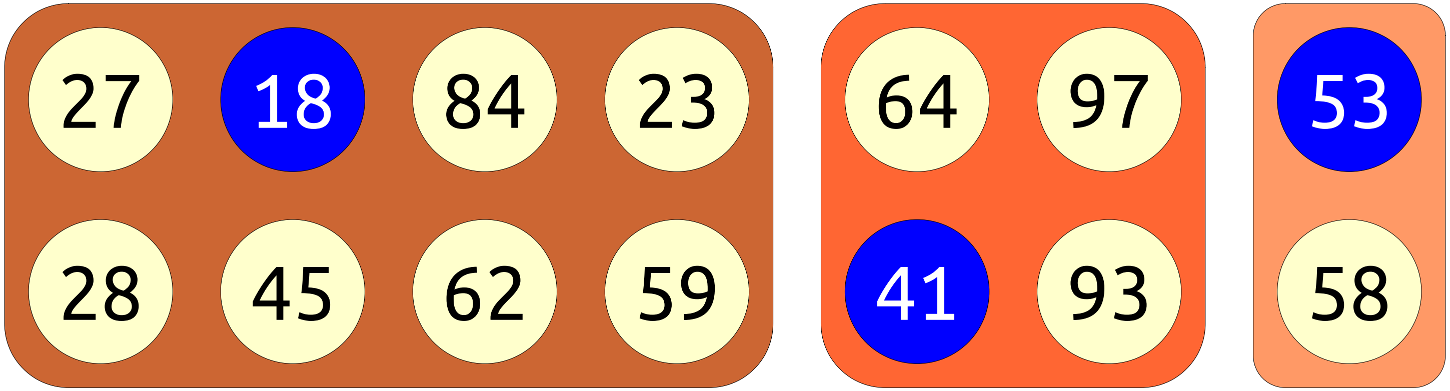
Time required:
O(log n) fuses.

Deleting the Minimum

- Our analogy with arithmetic breaks down when we try to remove the minimum element.
- After losing an element, the packet will not necessarily hold a number of elements that is a power of two.

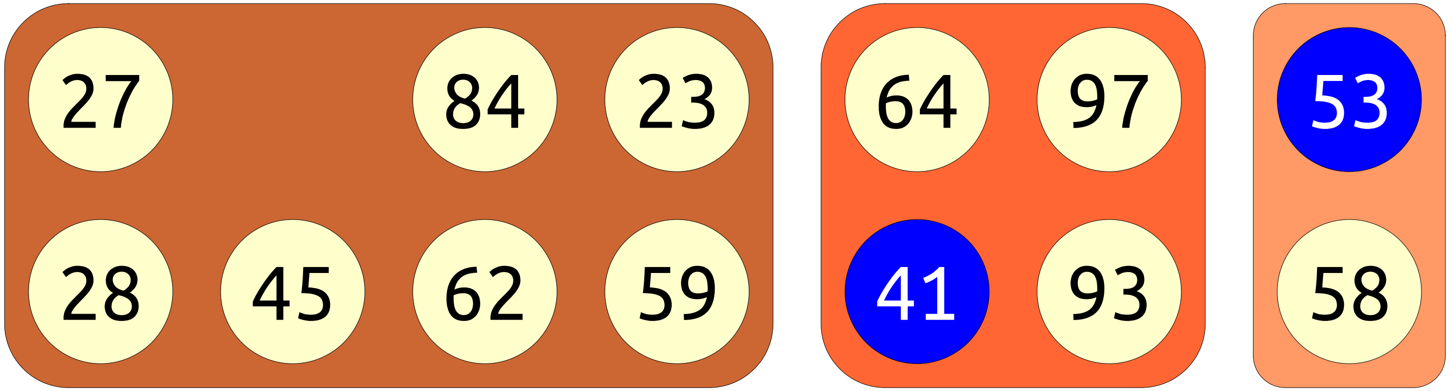
Deleting the Minimum

- Our analogy with arithmetic breaks down when we try to remove the minimum element.
- After losing an element, the packet will not necessarily hold a number of elements that is a power of two.



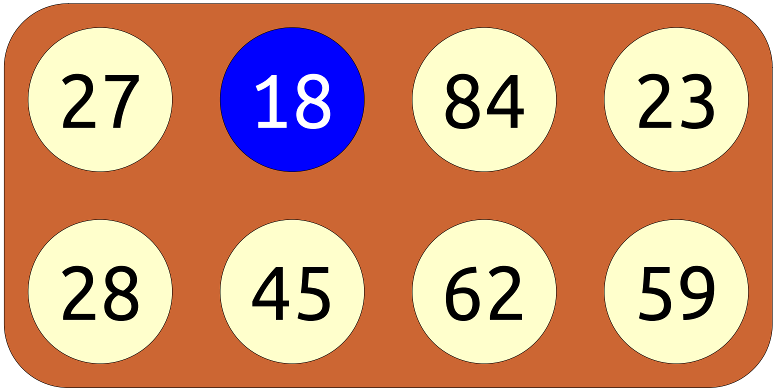
Deleting the Minimum

- Our analogy with arithmetic breaks down when we try to remove the minimum element.
- After losing an element, the packet will not necessarily hold a number of elements that is a power of two.



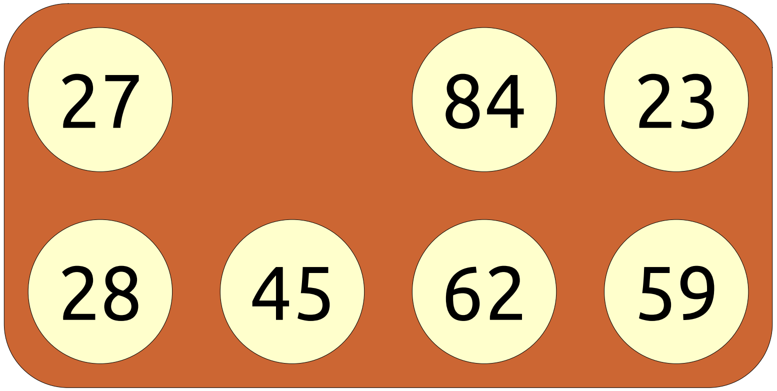
Deleting the Minimum

- If we have a packet with 2^k elements in it and remove a single element, we are left with $2^k – 1$ remaining elements.



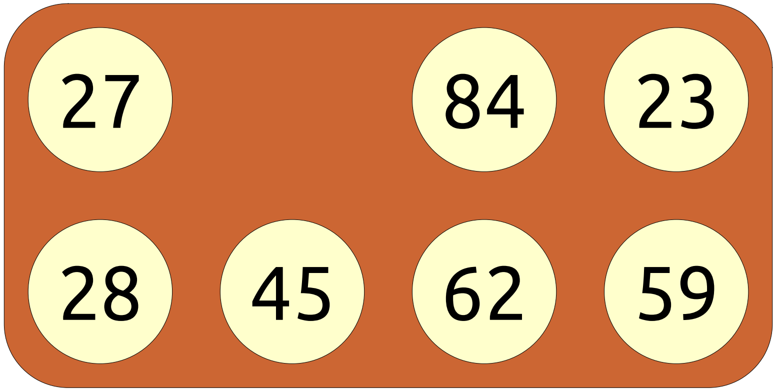
Deleting the Minimum

- If we have a packet with 2^k elements in it and remove a single element, we are left with $2^k - 1$ remaining elements.



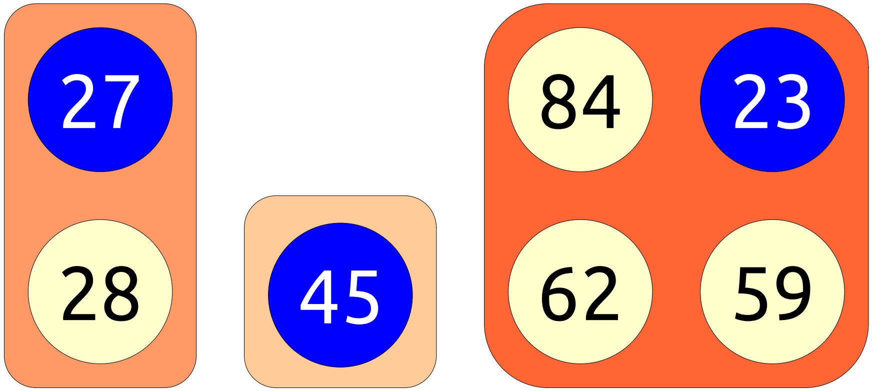
Deleting the Minimum

- If we have a packet with 2^k elements in it and remove a single element, we are left with $2^k - 1$ remaining elements.
- ***Fun fact***: $2^k - 1 = 2^0 + 2^1 + 2^2 + \ldots + 2^{k-1}$.
- ***Idea***: "Fracture" the packet into k smaller packets, then add them back in.



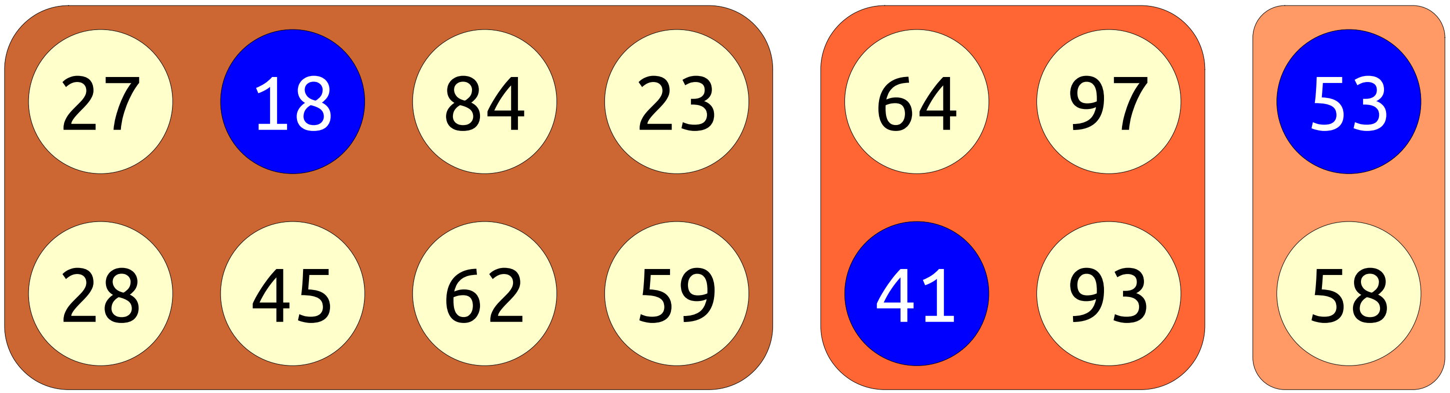
Deleting the Minimum

- If we have a packet with 2^k elements in it and remove a single element, we are left with $2^k – 1$ remaining elements.
- ***Fun fact***: $2^k – 1 = 2^0 + 2^1 + 2^2 + \ldots + 2^{k-1}$.
- ***Idea***: “Fracture” the packet into k smaller packets, then add them back in.



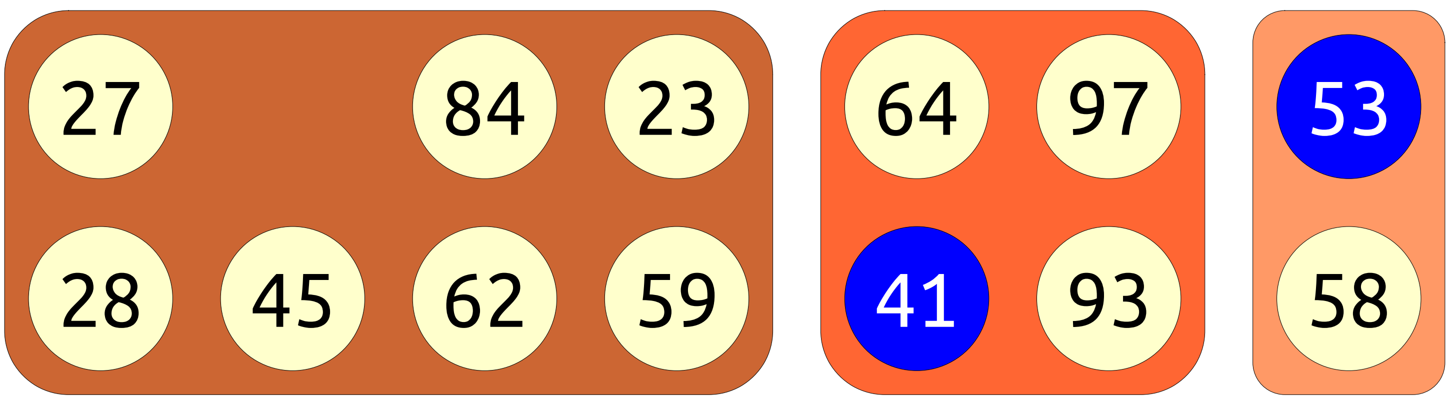
Fracturing Packets

- We can ***extract-min*** by fracturing the packet containing the minimum and adding the fragments back in.



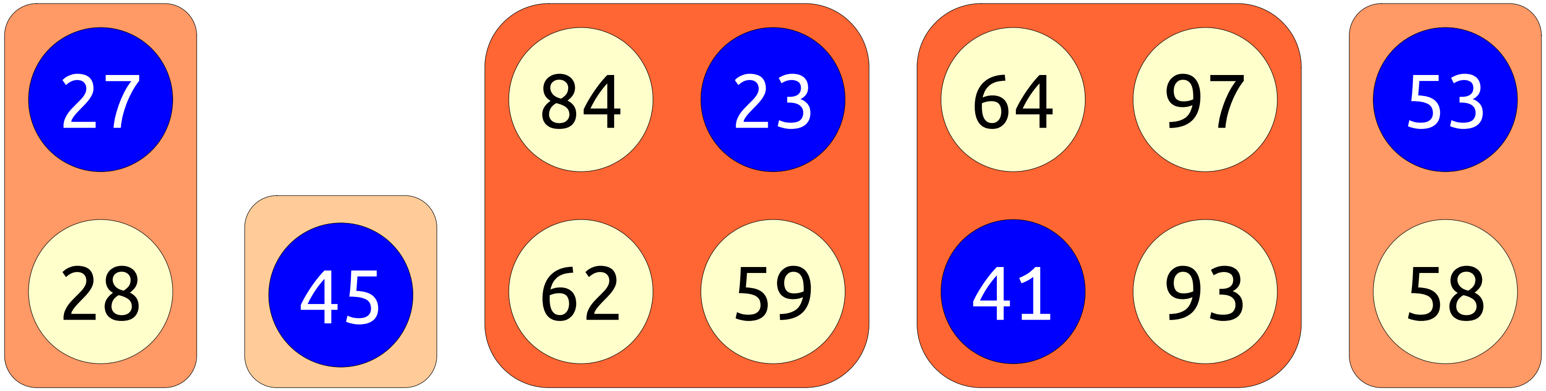
Fracturing Packets

- We can ***extract-min*** by fracturing the packet containing the minimum and adding the fragments back in.



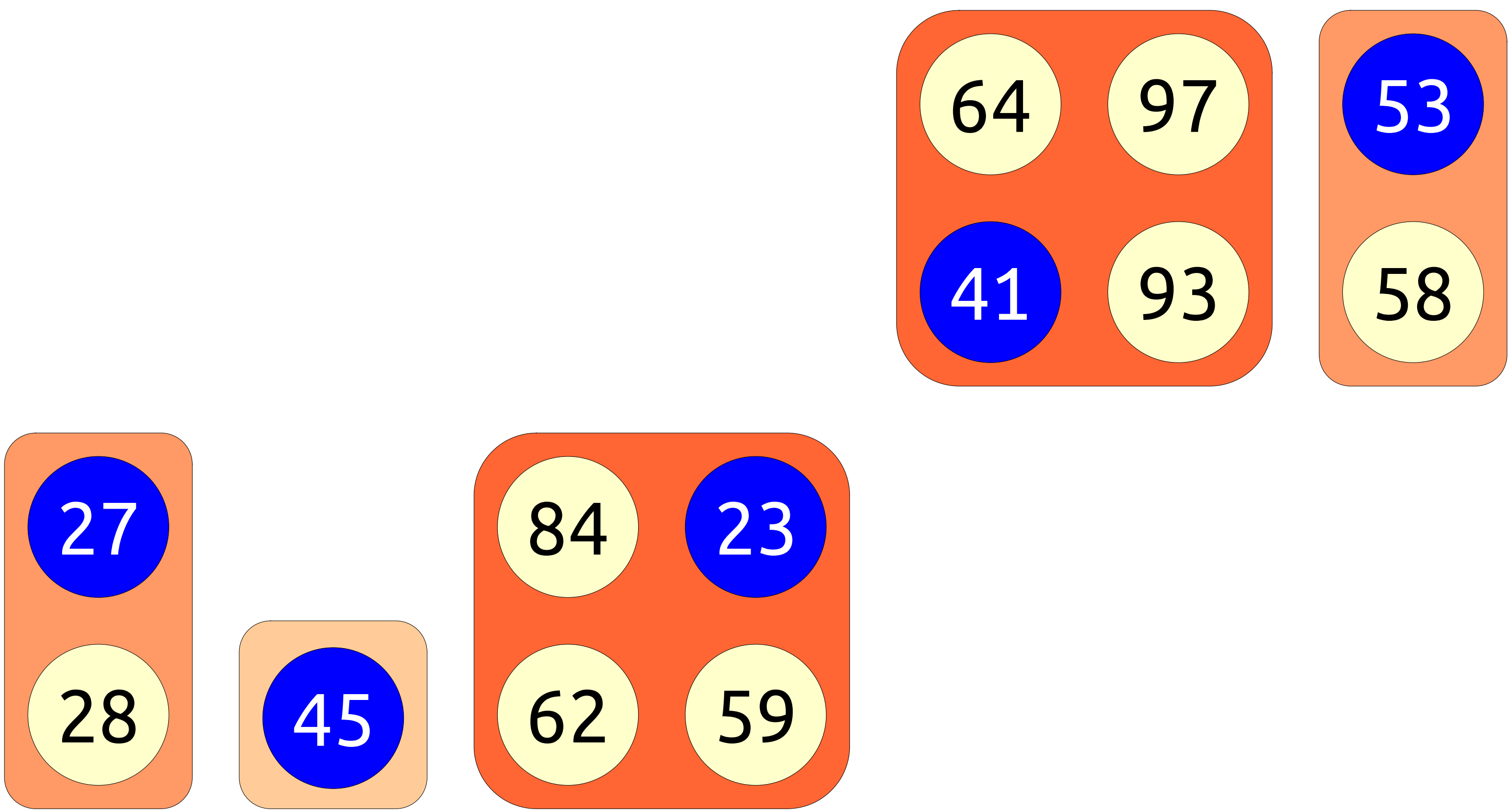
Fracturing Packets

- We can ***extract-min*** by fracturing the packet containing the minimum and adding the fragments back in.



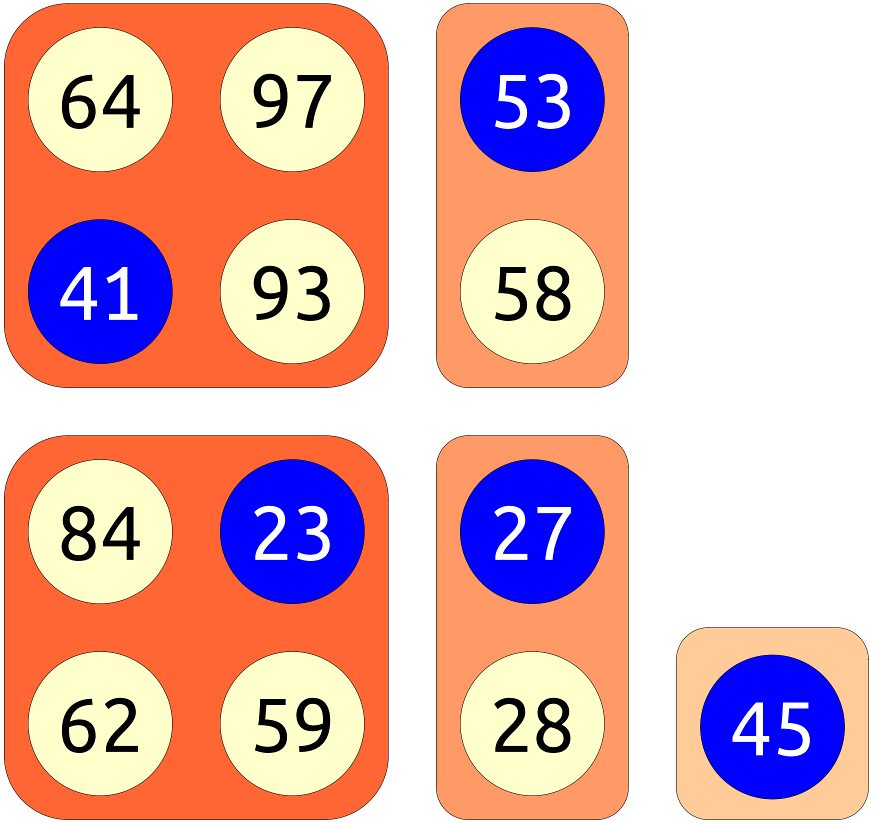
Fracturing Packets

- We can ***extract-min*** by fracturing the packet containing the minimum and adding the fragments back in.



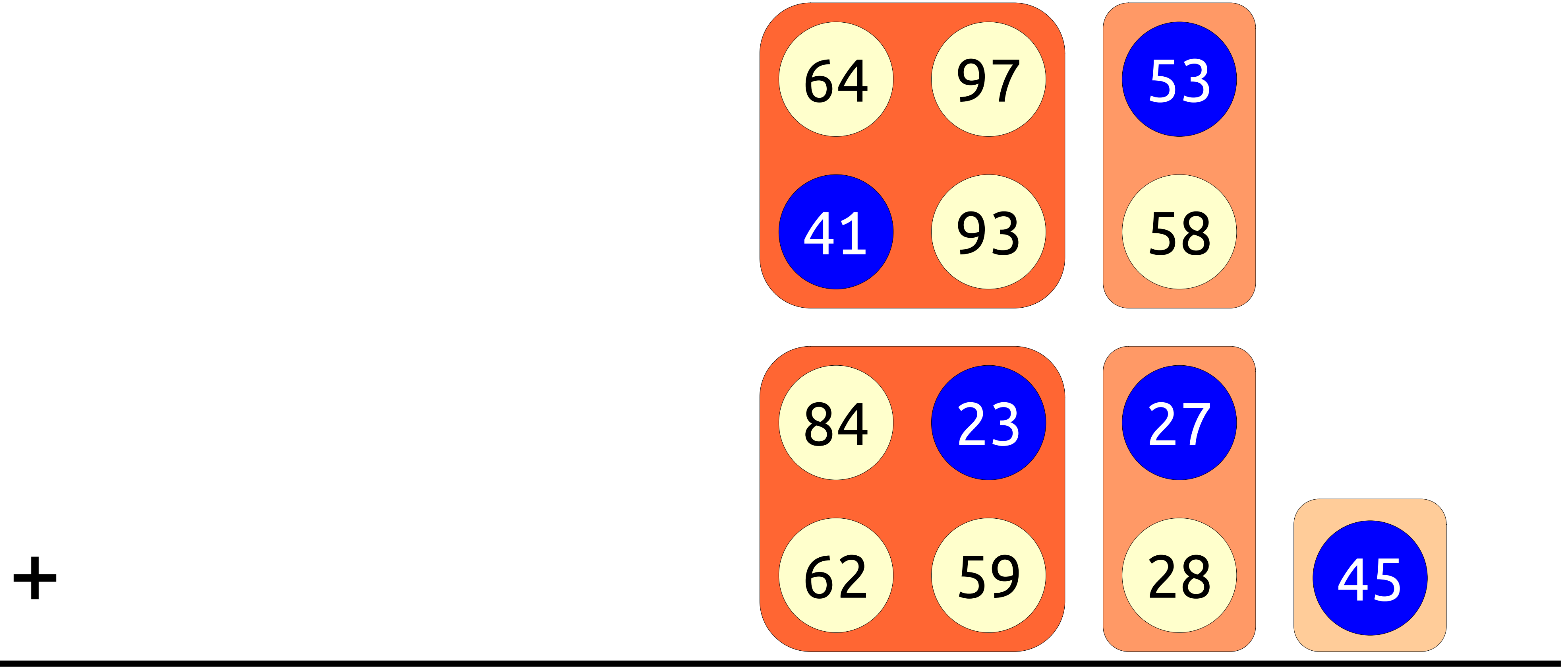
Fracturing Packets

- We can ***extract-min*** by fracturing the packet containing the minimum and adding the fragments back in.



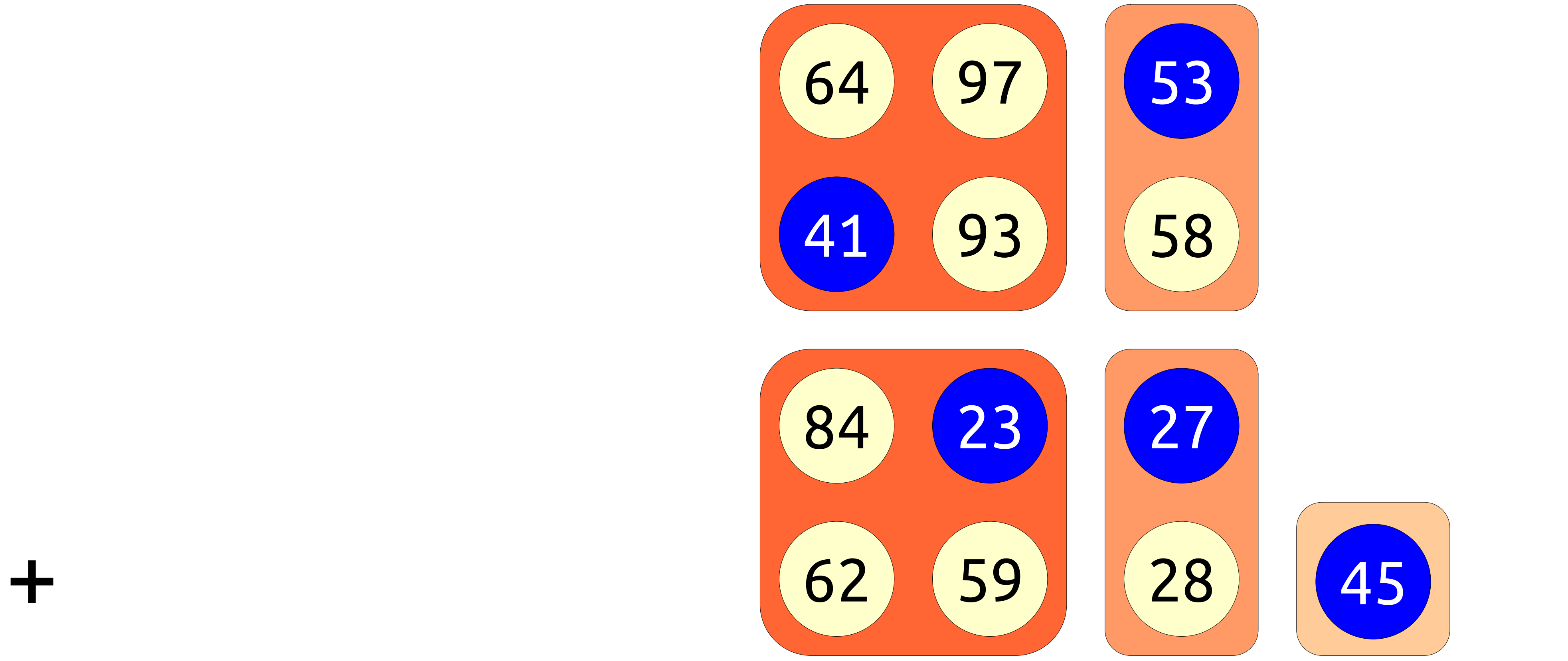
Fracturing Packets

- We can ***extract-min*** by fracturing the packet containing the minimum and adding the fragments back in.



Fracturing Packets

- We can ***extract-min*** by fracturing the packet containing the minimum and adding the fragments back in.
- Runtime is O(log n) fuses in ***meld***, plus fracture cost.



Building a Priority Queue

- What properties must our packets have?
  - Size is a power of two.
  - Can efficiently fuse packets of the same size.
  - Can efficiently find the minimum element of each packet.
  - Can efficiently "fracture" a packet of 2^k nodes into packets of 2^0, 2^1, 2^2, 2^3, …, 2^{k-1} nodes.
- ***Question:*** How can we represent our packets to support the above operations efficiently?

Propose a solution!

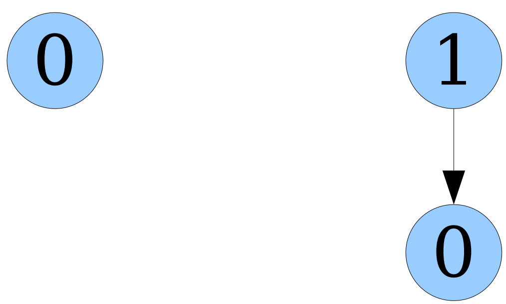
https://cs166.stanford.edu/pollev

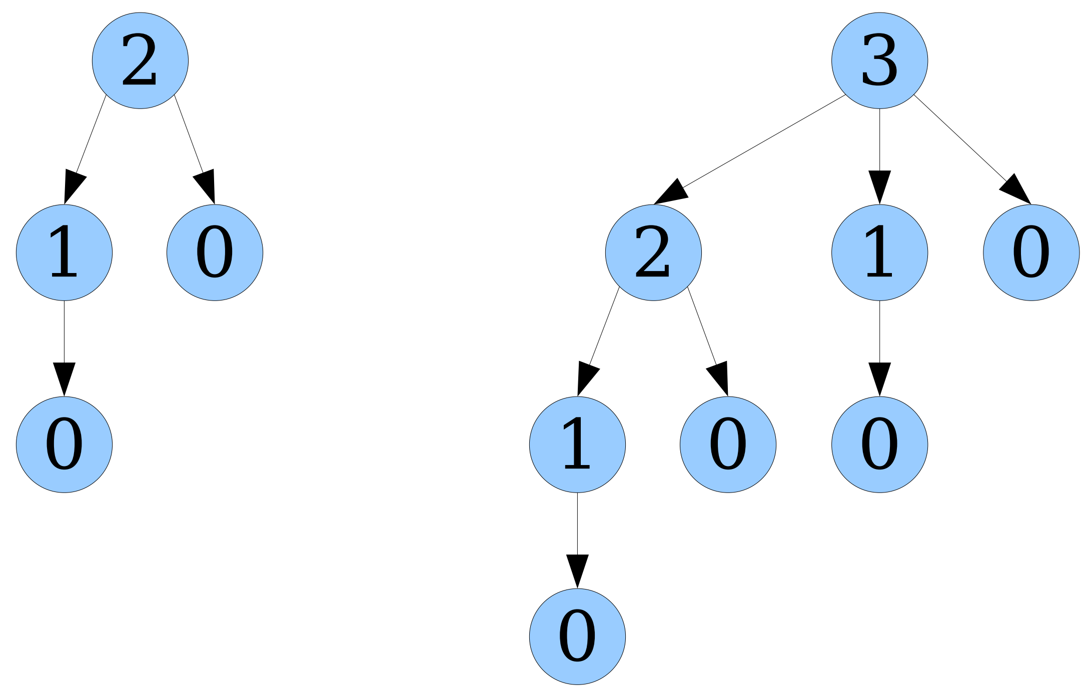
Binomial Trees

- A ***binomial tree of order k*** is a type of tree recursively defined as follows:

A binomial tree of order k is a single node whose children are binomial trees of order 0, 1, 2, …, k – 1.

- Here are the first few binomial trees:





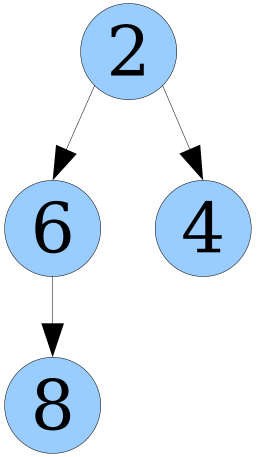
Why are these called binomial trees? Look across the layers of these trees and see if you notice anything!

Binomial Trees

- What properties must our packets have?
  - Size must be a power of two.
  - Can efficiently fuse packets of the same size.
  - Can efficiently find the minimum element of each packet.
  - Can efficiently “fracture” a packet of 2^k nodes into packets of 2^0, 2^1, 2^2, 2^3, …, 2^{k-1} nodes.

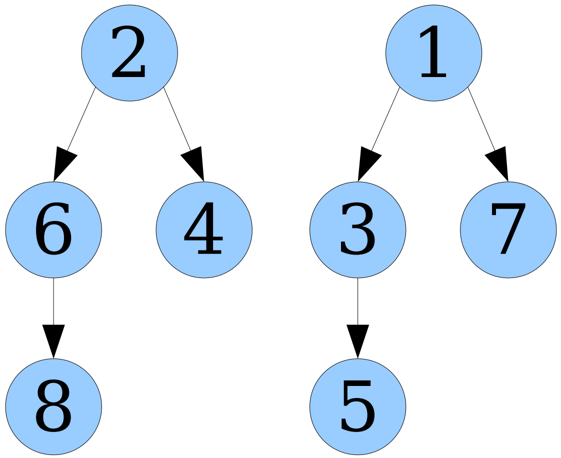
Binomial Trees

- What properties must our packets have?
  - Size must be a power of two. √
  - Can efficiently fuse packets of the same size.
  - Can efficiently find the minimum element of each packet.
  - Can efficiently "fracture" a packet of 2^k nodes into packets of 2^0, 2^1, 2^2, 2^3, …, 2^{k-1} nodes.



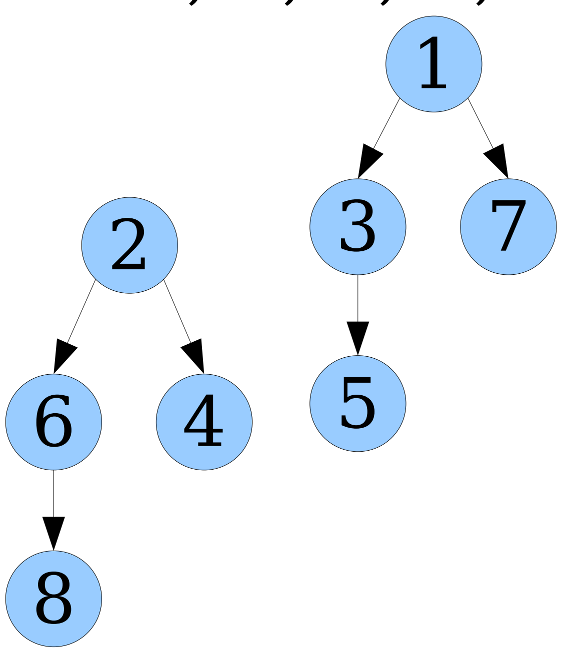
Binomial Trees

- What properties must our packets have?
  - Size must be a power of two. √
  - Can efficiently fuse packets of the same size.
  - Can efficiently find the minimum element of each packet.
  - Can efficiently "fracture" a packet of 2^k nodes into packets of 2^0, 2^1, 2^2, 2^3, …, 2^{k-1} nodes.



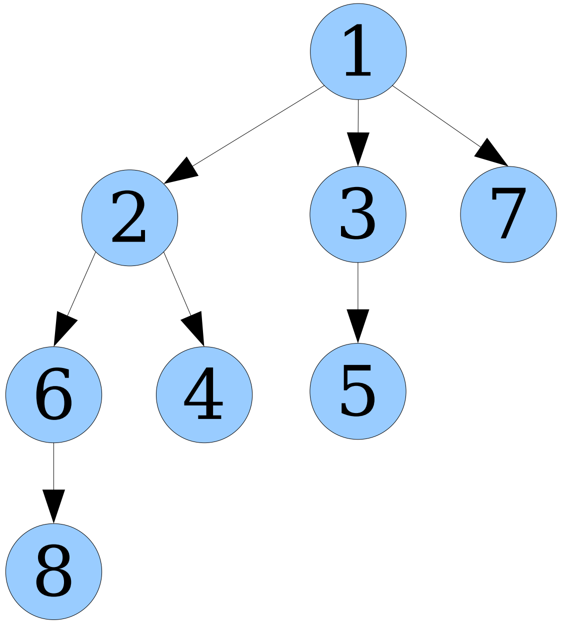
Binomial Trees

- What properties must our packets have?
  - Size must be a power of two. √
  - Can efficiently fuse packets of the same size.
  - Can efficiently find the minimum element of each packet.
  - Can efficiently "fracture" a packet of 2^k nodes into packets of 2^0, 2^1, 2^2, 2^3, …, 2^{k-1} nodes.



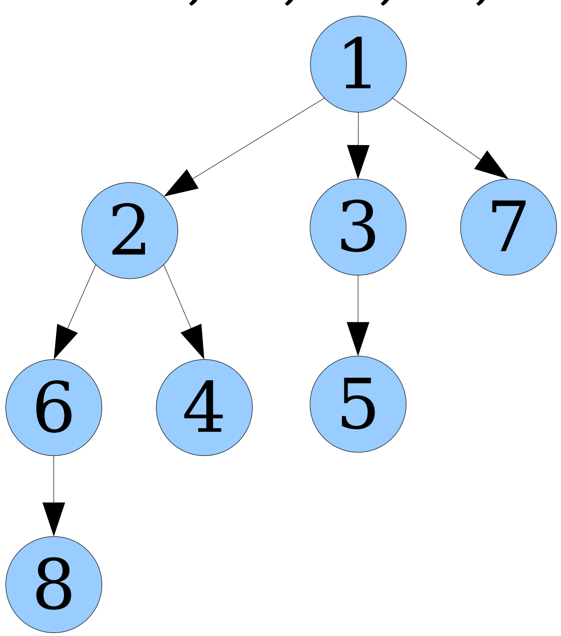
Binomial Trees

- What properties must our packets have?
  - Size must be a power of two. √
  - Can efficiently fuse packets of the same size.
  - Can efficiently find the minimum element of each packet.
  - Can efficiently "fracture" a packet of 2^k nodes into packets of 2^0, 2^1, 2^2, 2^3, …, 2^{k-1} nodes.



Binomial Trees

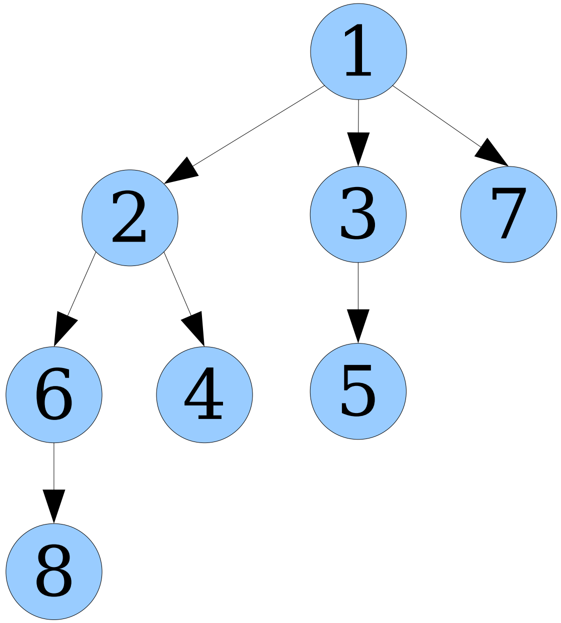
- What properties must our packets have?
  - Size must be a power of two. √
  - Can efficiently fuse packets of the same size.
  - Can efficiently find the minimum element of each packet.
  - Can efficiently "fracture" a packet of 2^k nodes into packets of 2^0, 2^1, 2^2, 2^3, …, 2^{k-1} nodes.



Make the binomial tree with the larger root the first child of the tree with the smaller root.

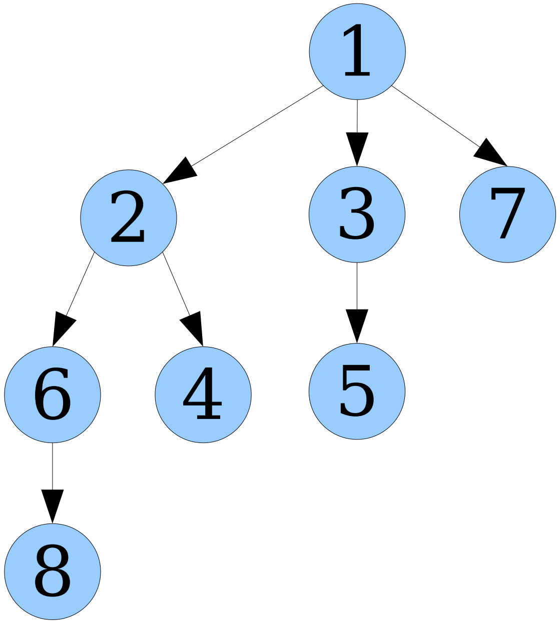
Binomial Trees

- What properties must our packets have?
  - Size must be a power of two. √
  - Can efficiently fuse packets of the same size. √
  - Can efficiently find the minimum element of each packet.
  - Can efficiently "fracture" a packet of 2^k nodes into packets of 2^0, 2^1, 2^2, 2^3, …, 2^{k-1} nodes.



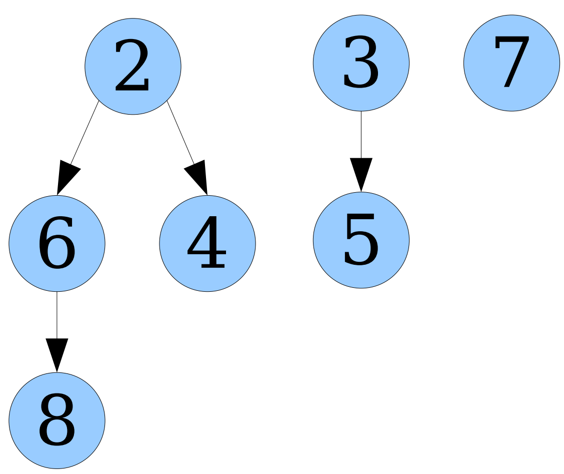
Binomial Trees

- What properties must our packets have?
  - Size must be a power of two. √
  - Can efficiently fuse packets of the same size. √
  - Can efficiently find the minimum element of each packet. √
  - Can efficiently "fracture" a packet of 2^k nodes into packets of $2^0, 2^1, 2^2, 2^3, \ldots, 2^{k-1}$ nodes.



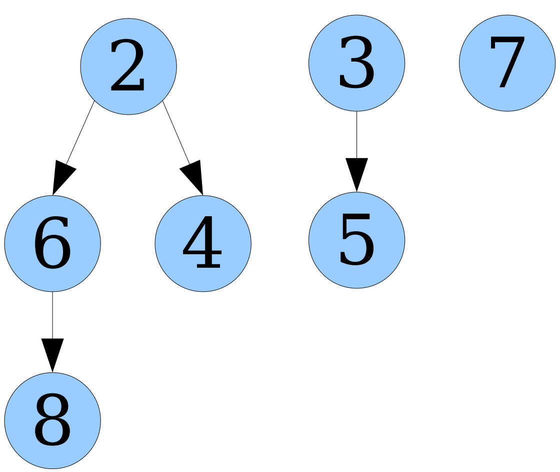
Binomial Trees

- What properties must our packets have?
  - Size must be a power of two. √
  - Can efficiently fuse packets of the same size. √
  - Can efficiently find the minimum element of each packet. √
  - Can efficiently "fracture" a packet of 2^k nodes into packets of 2^0, 2^1, 2^2, 2^3, …, 2^{k-1} nodes.



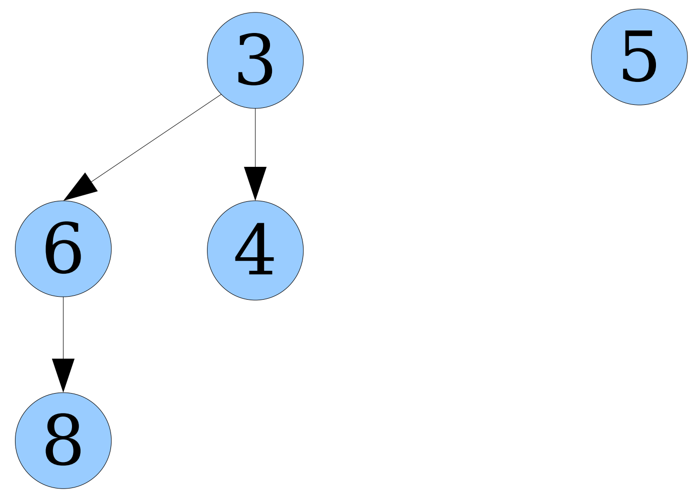
Binomial Trees

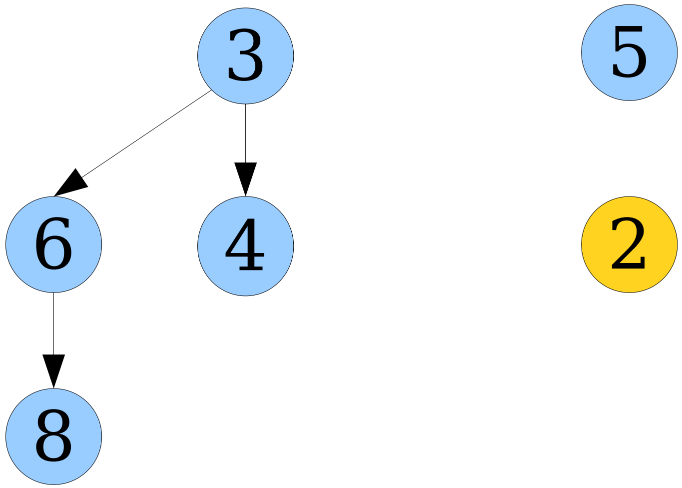
- What properties must our packets have?
  - Size must be a power of two. √
  - Can efficiently fuse packets of the same size. √
  - Can efficiently find the minimum element of each packet. √
  - Can efficiently "fracture" a packet of 2^k nodes into packets of 2^0, 2^1, 2^2, 2^3, …, 2^{k-1} nodes. √

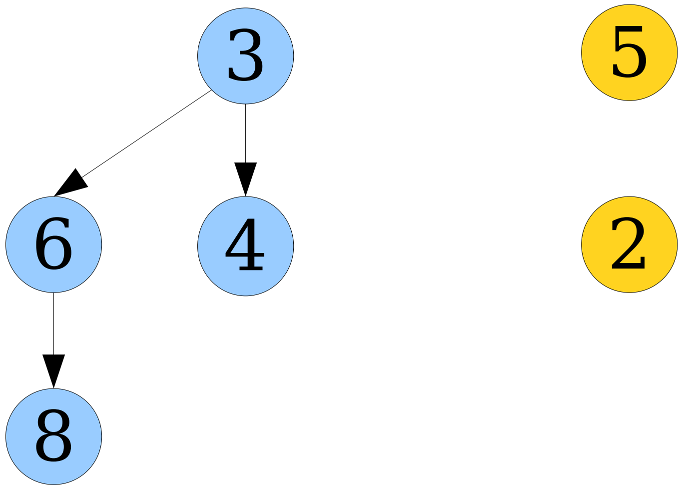


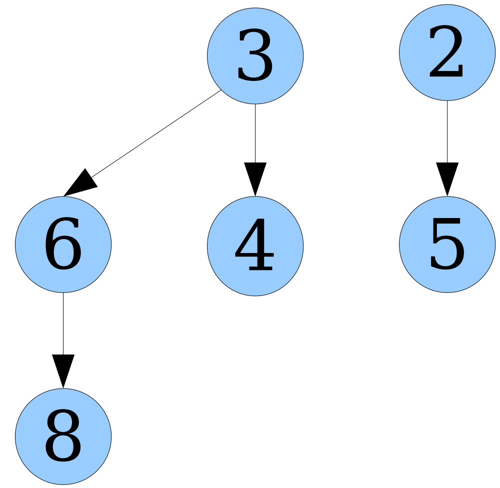
The Binomial Heap

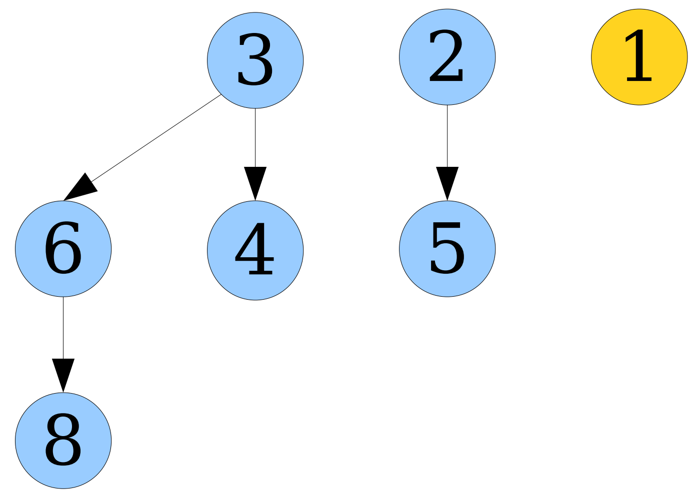
- A ***binomial heap*** is a collection of heap-ordered binomial trees stored in ascending order of size.
- Operations defined as follows:
  - ***meld***(pq_1, pq_2): Use addition to combine all the trees.
    - Fuses O(log *n* + log *m*) trees. Cost: O(log *n* + log *m*). Here, assume one binomial heap has *n* nodes, the other *m*.
  - *pq*.***enqueue***(*v*, *k*): Meld *pq* and a singleton heap of (*v*, *k*).
    - Total time: O(log *n*).
  - *pq*.***find-min***(): Find the minimum of all tree roots.
    - Total time: O(log *n*).
  - *pq*.***extract-min***(): Find the min, delete the tree root, then meld together the queue and the exposed children.
    - Total time: O(log *n*).

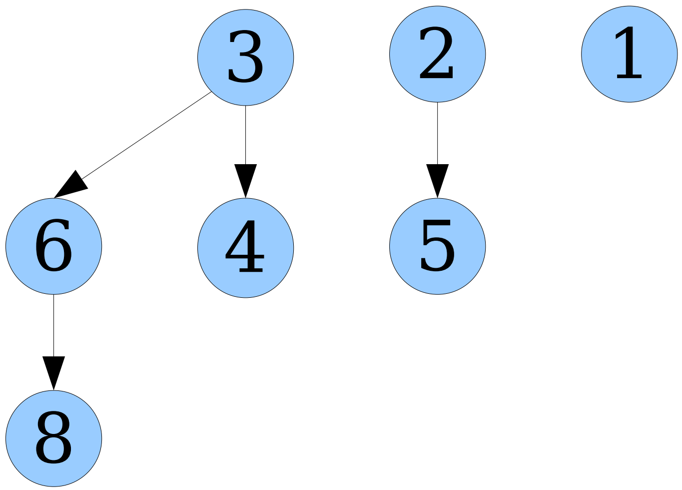

3
5
6
4
8

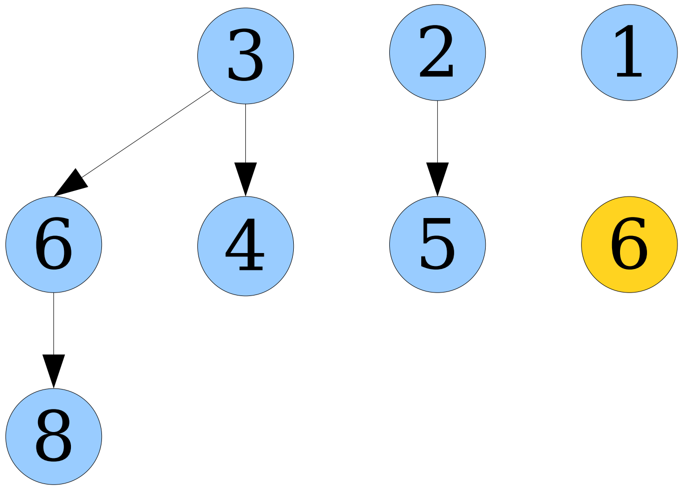

3
5
6
4
2
8

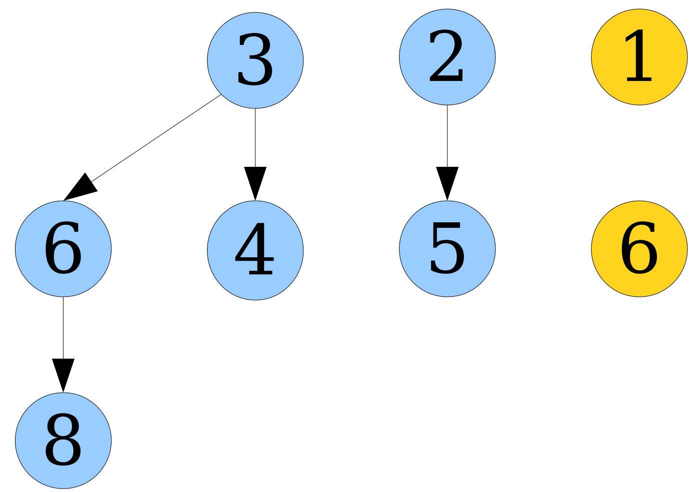

3
5
6
4
2
8

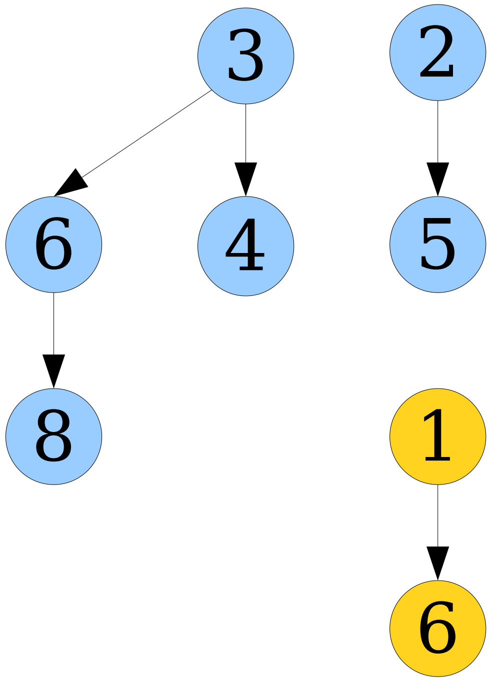

3
2
6
4
5
8

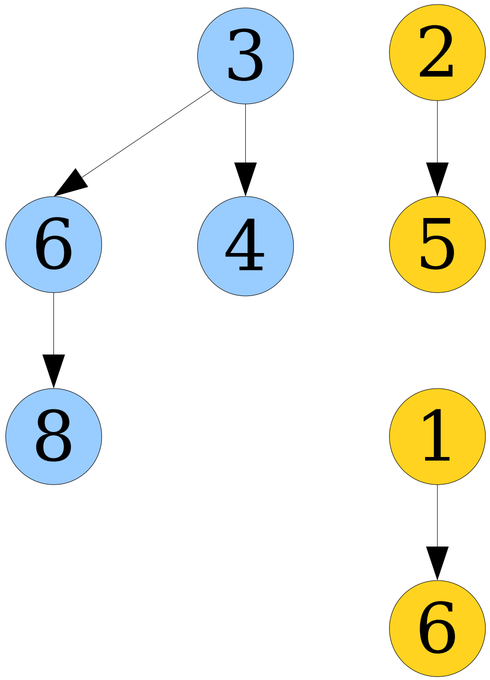

3
2
1
6
4
5
8

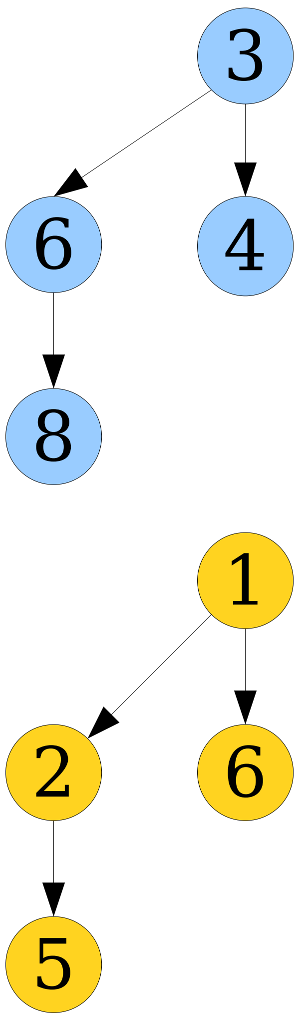

3
2
1
6
4
5
8

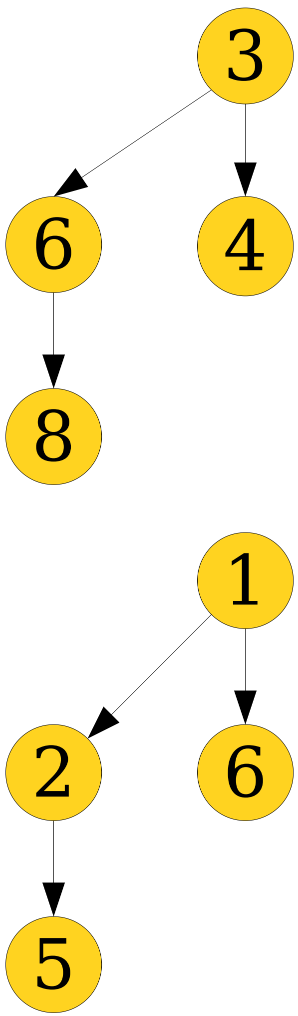

3
2
1
6
4
5
6
8

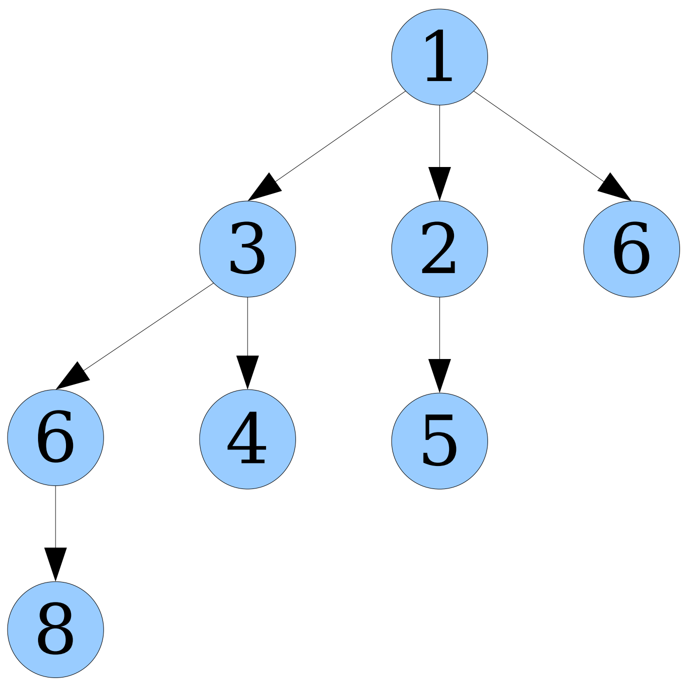

3
2
1
6
4
5
6
8

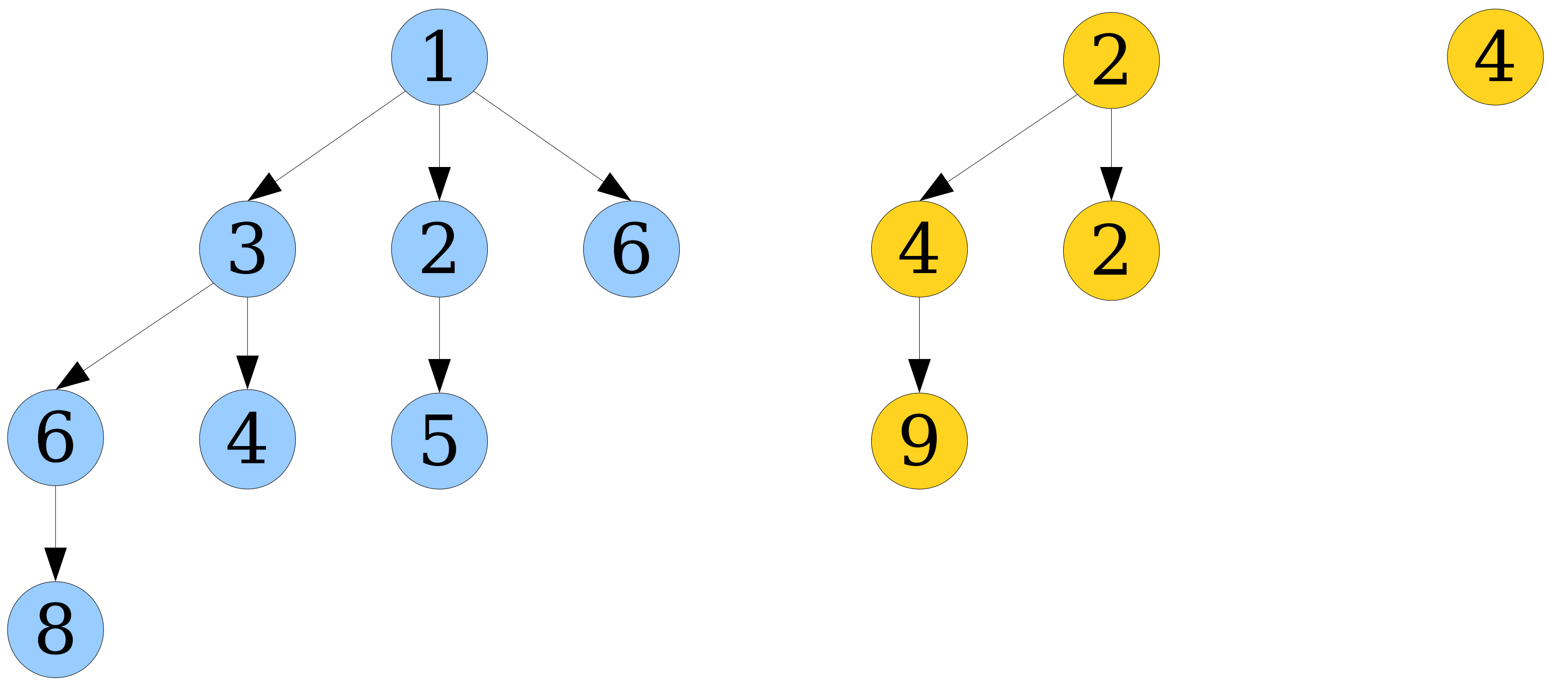

3
2
6
4
5
8
1
6

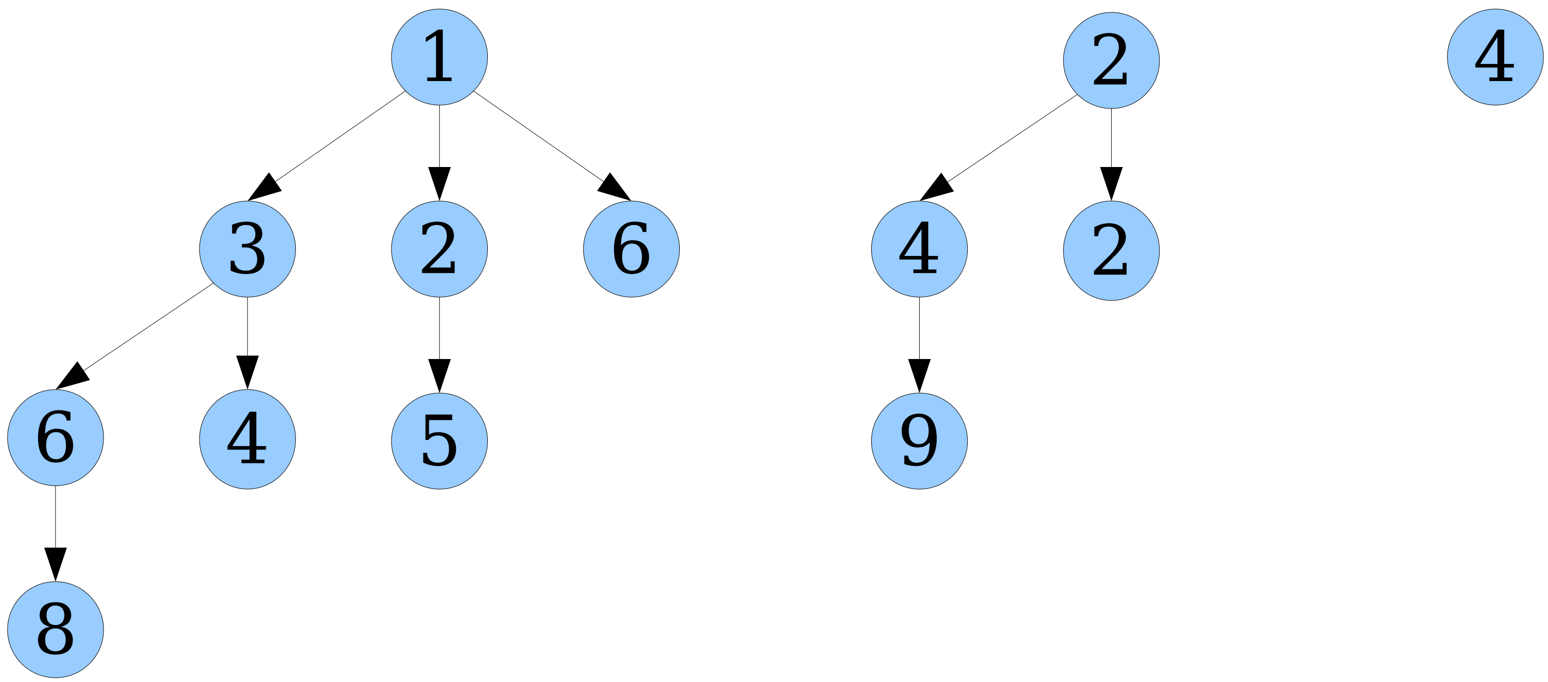

3
2
6
4
5
8
1
6

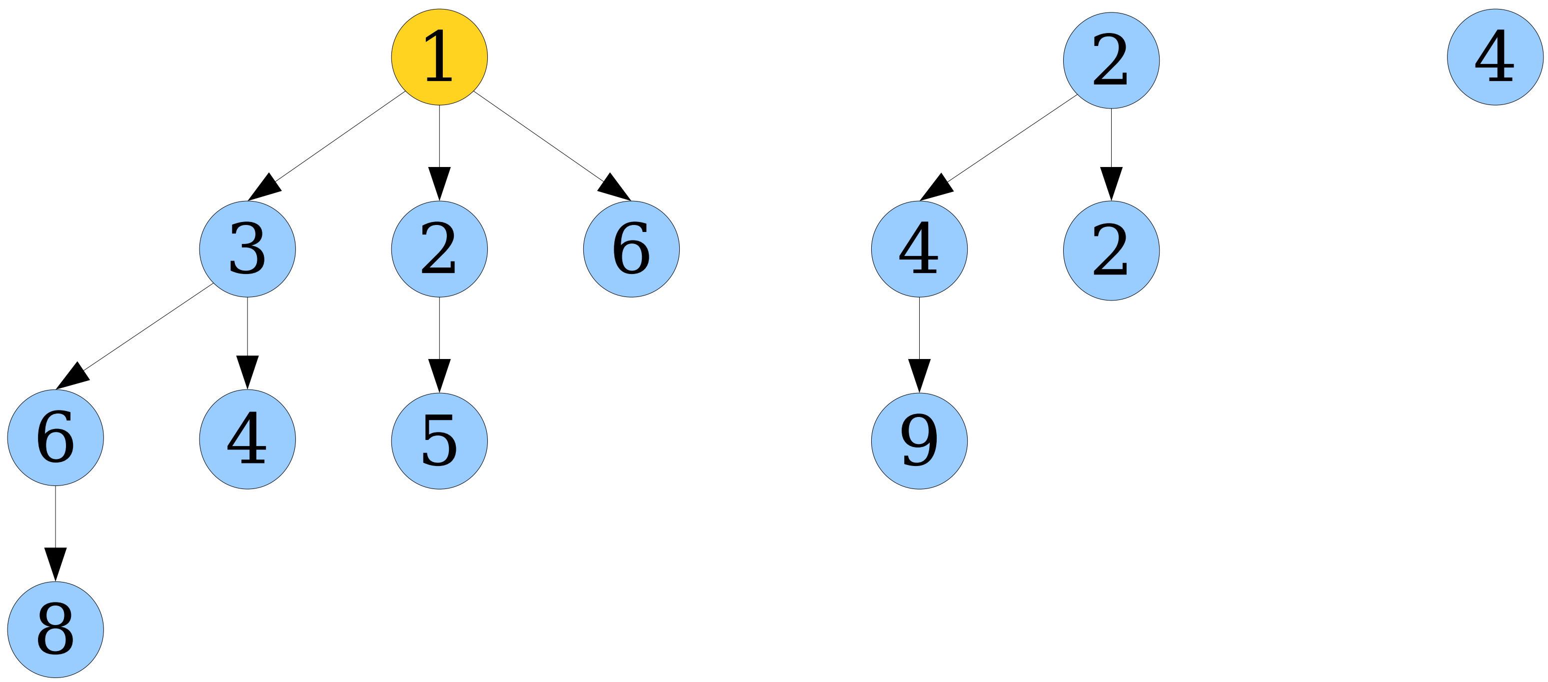

3
6
4
8
1
2
6
5

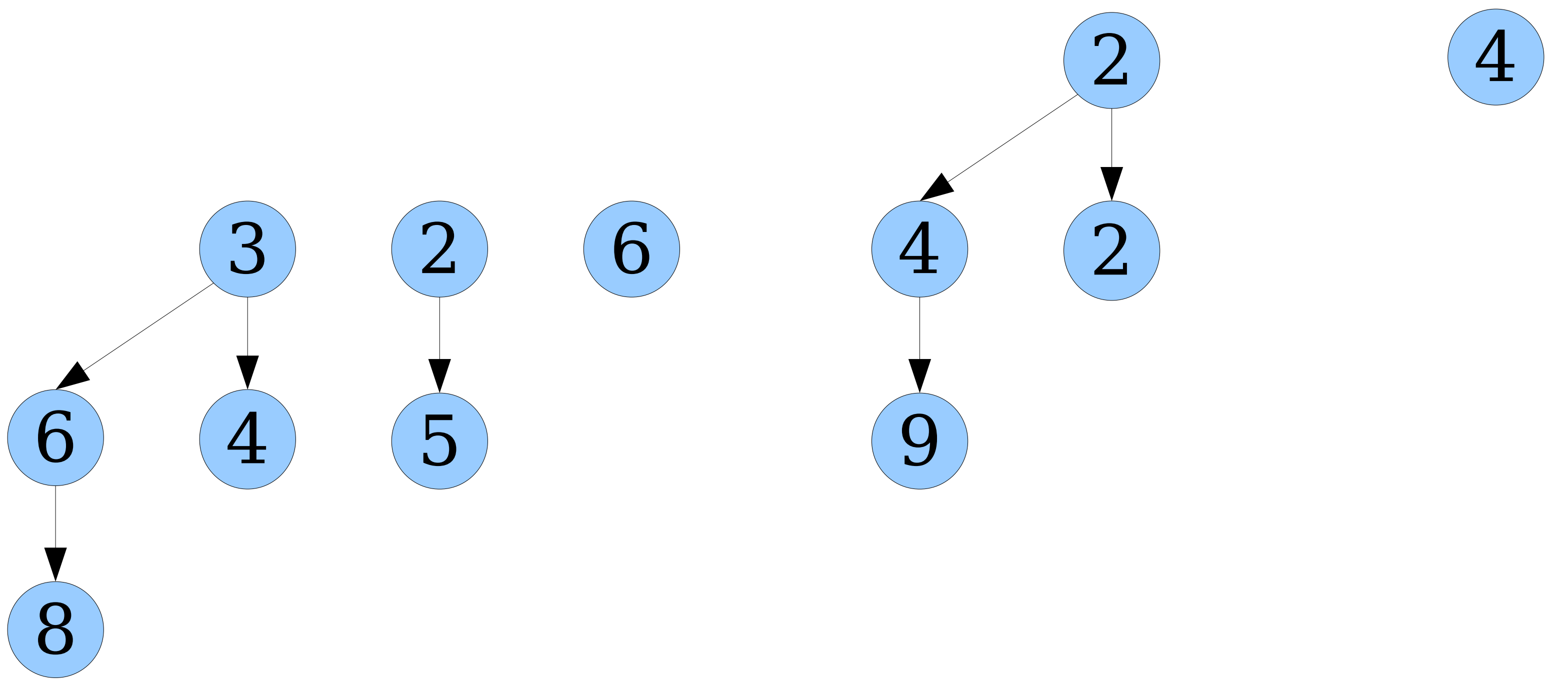

3
6
4
8
1
2
6
5

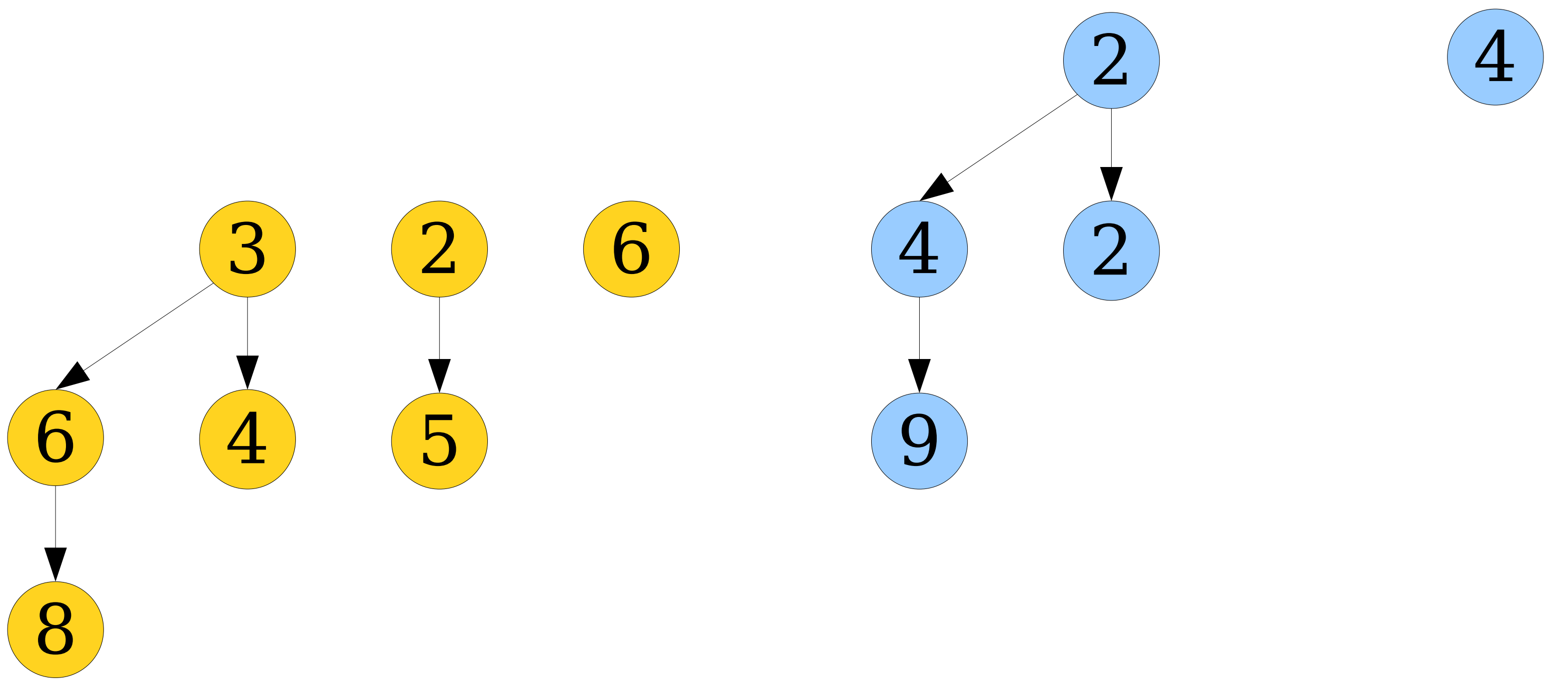

1
3
2
6
6
4
5
8


1
3
2
6
6
4
5
8
2
4
2
9
4

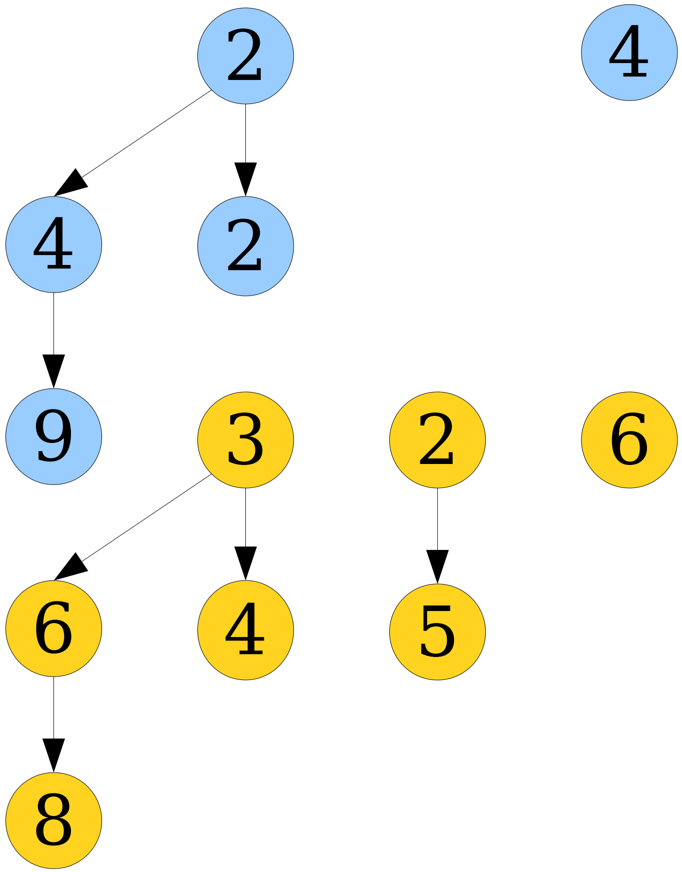

1
3
2
6
6
4
5
8
2
4
2
9
4

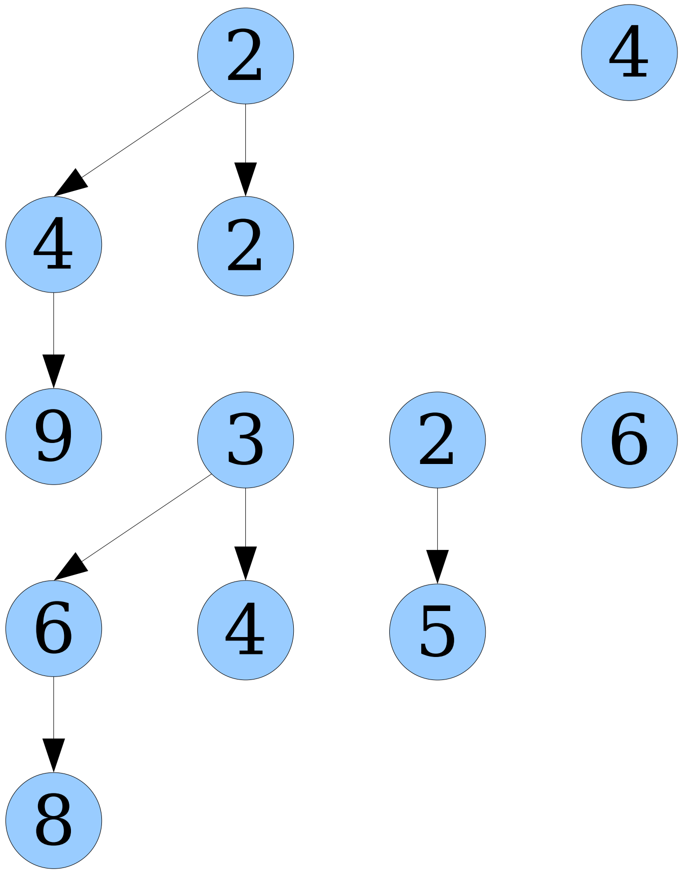

1
3
2
6
6
4
5
8
2
4
2
9
4

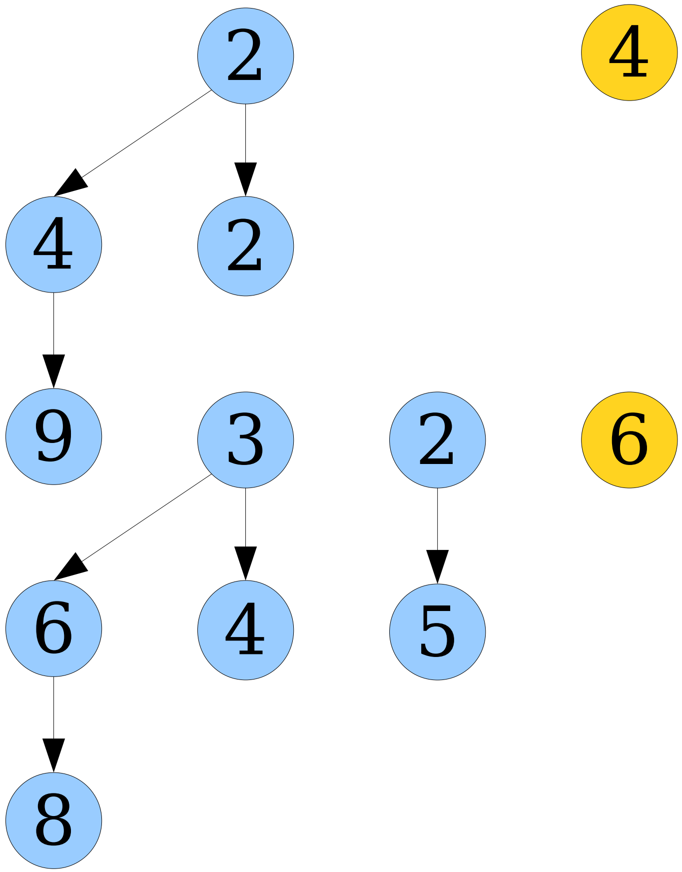

3
6
4
8
2
5
6
2
4
2
9
4

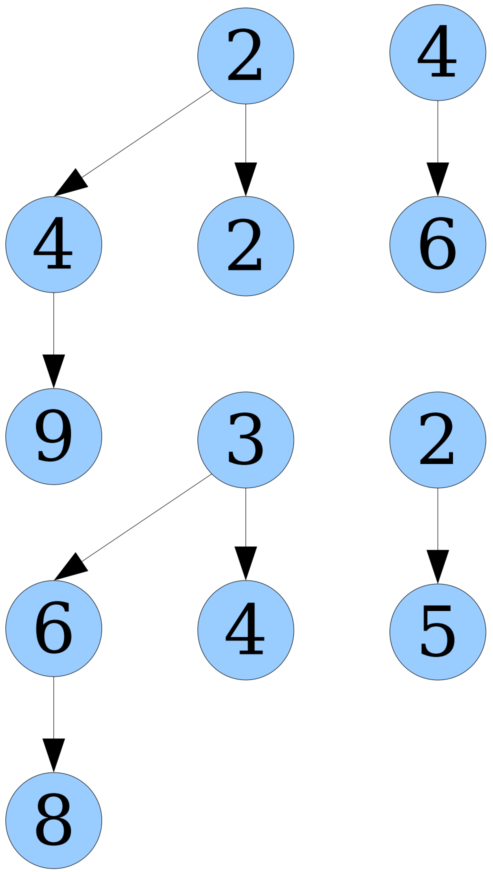


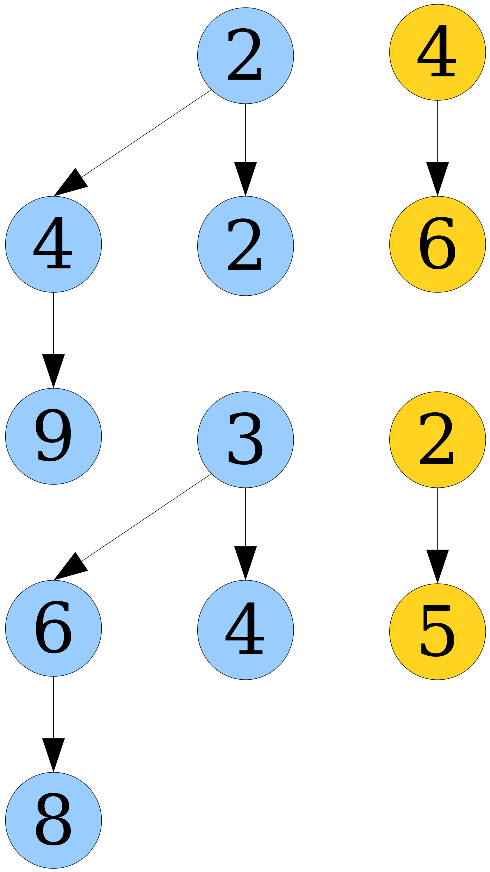
3
2
6
6
4
5
8
2
4
4
2
9


2
4
4
2
9
3
2
6
6
4
5
8

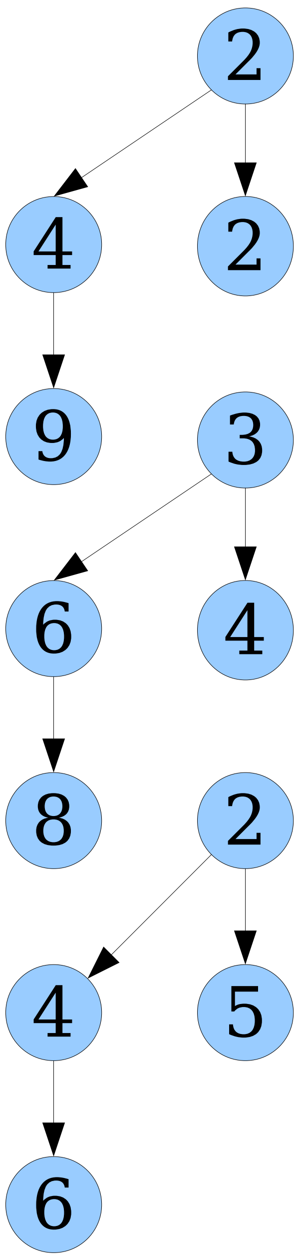

2
4
4
2
9
3
2
6
6
4
5
8

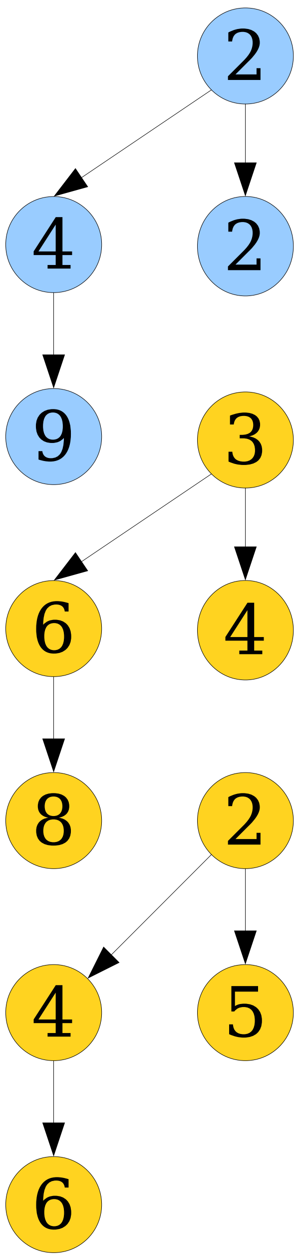

2
4
4
2
9
3
2
6
6
4
5
8

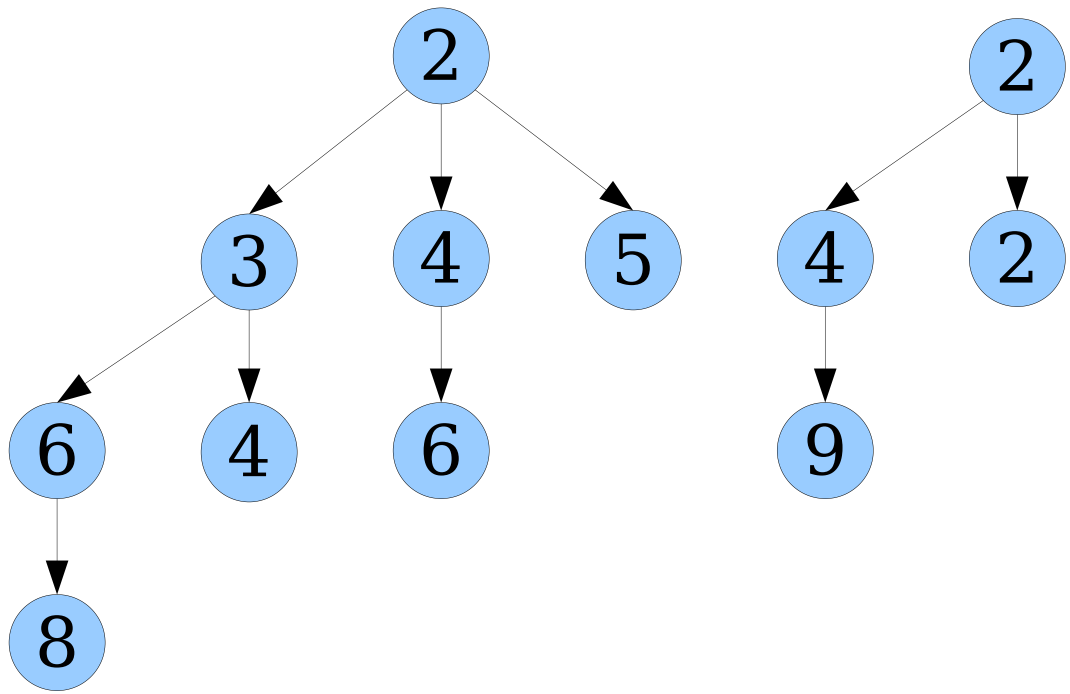

2
4
4
2
6
9
3
2
6
4
5
8



2
4
4
2
6
9
3
2
6
4
5
8


2
4
2
9
3
6
4
8
2
4
5
6


2
4
2
9
3
6
4
8
2
4
5
6

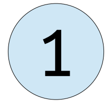

2
3
4
5
6
4
6
8
2
4
2
9

Do this, and tell us what the roots of the resulting trees are.

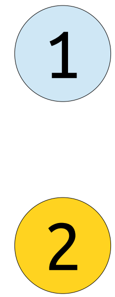
Answer at
https://cs166.stanford.edu/pollev

Draw what happens if we ***enqueue*** the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 into a binomial heap.

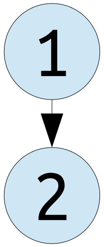
Draw what happens if we ***enqueue*** the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 into a binomial heap.



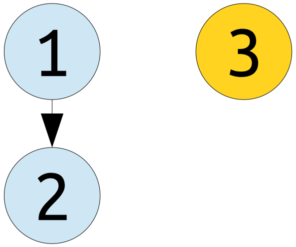
Draw what happens if we ***enqueue*** the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 into a binomial heap.



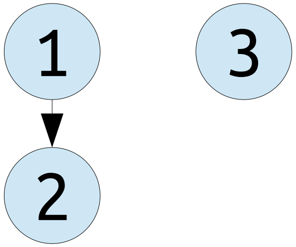
Draw what happens if we ***enqueue*** the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 into a binomial heap.



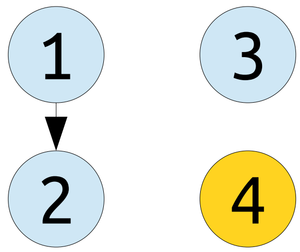
Draw what happens if we ***enqueue*** the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 into a binomial heap.



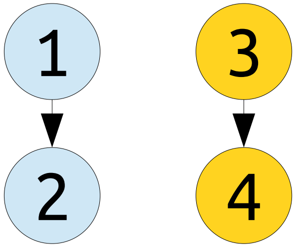
Draw what happens if we ***enqueue*** the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 into a binomial heap.



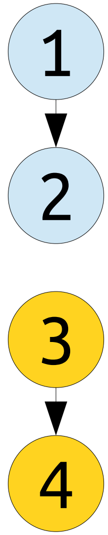
Draw what happens if we ***enqueue*** the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 into a binomial heap.



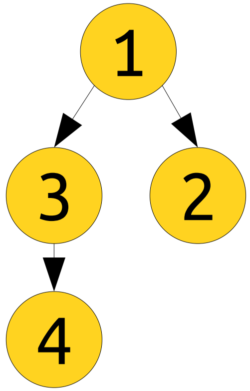
Draw what happens if we ***enqueue*** the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 into a binomial heap.



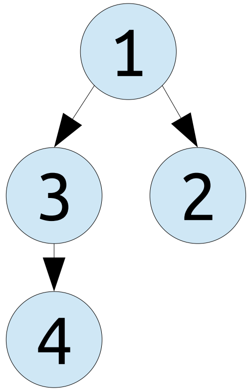
Draw what happens if we ***enqueue*** the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 into a binomial heap.



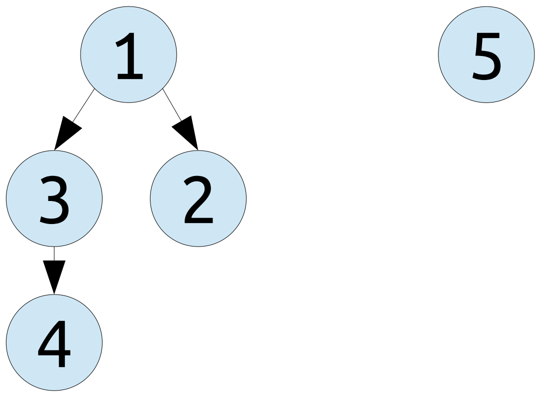
Draw what happens if we ***enqueue*** the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 into a binomial heap.



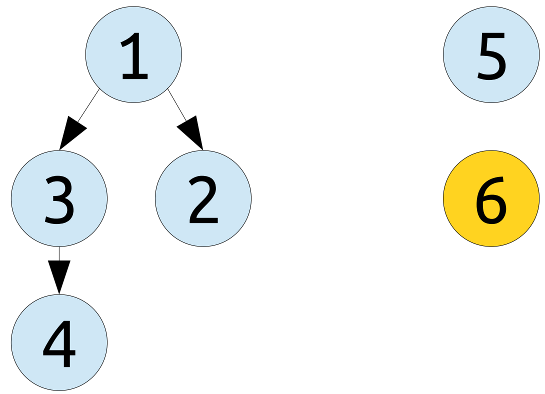
Draw what happens if we ***enqueue*** the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 into a binomial heap.



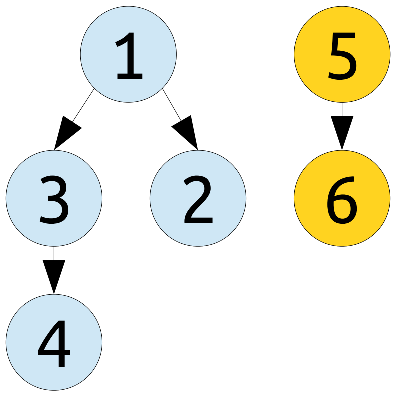
Draw what happens if we ***enqueue*** the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 into a binomial heap.



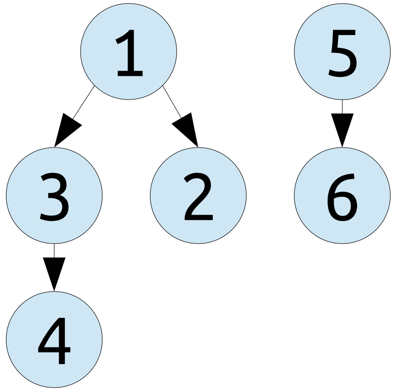
Draw what happens if we ***enqueue*** the numbers
1, 2, 3, 4, 5, 6, 7, 8, and 9 into a binomial heap.



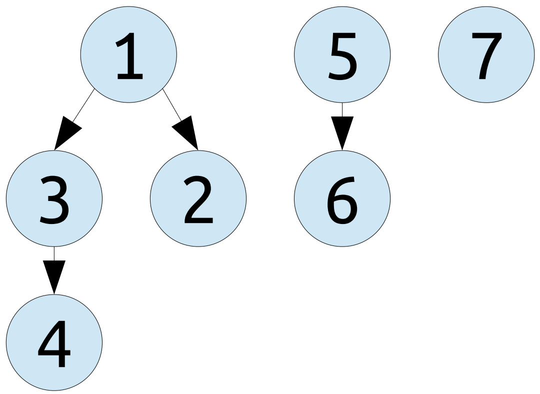
Draw what happens if we ***enqueue*** the numbers
1, 2, 3, 4, 5, 6, 7, 8, and 9 into a binomial heap.



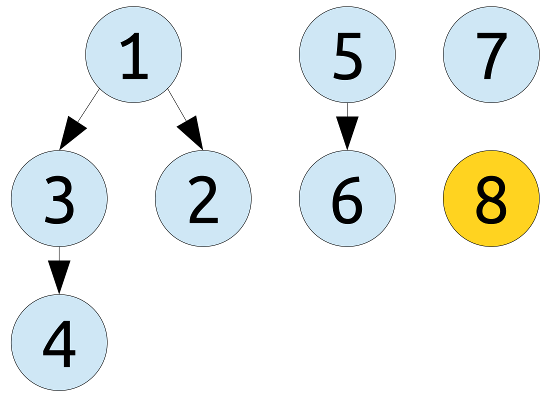
Draw what happens if we ***enqueue*** the numbers
1, 2, 3, 4, 5, 6, 7, 8, and 9 into a binomial heap.



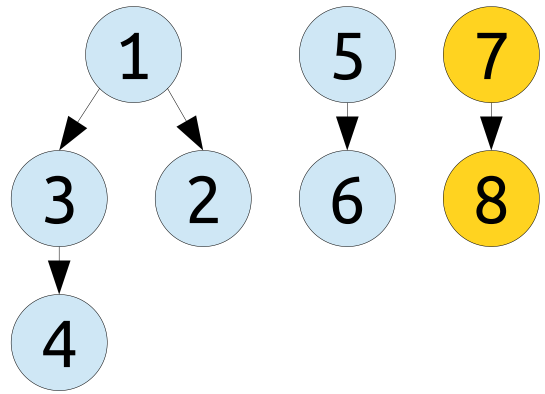
Draw what happens if we ***enqueue*** the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 into a binomial heap.



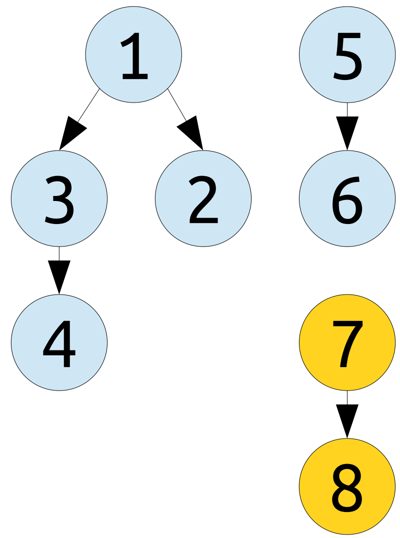
Draw what happens if we ***enqueue*** the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 into a binomial heap.



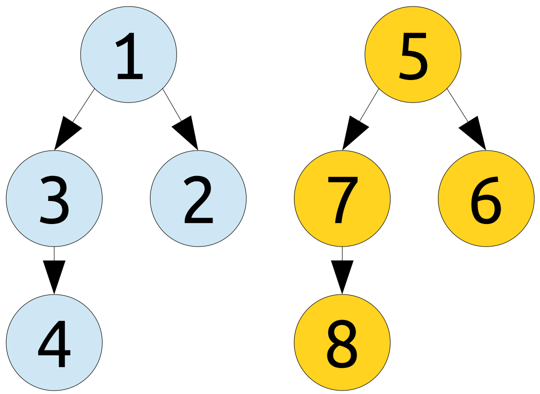
Draw what happens if we ***enqueue*** the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 into a binomial heap.



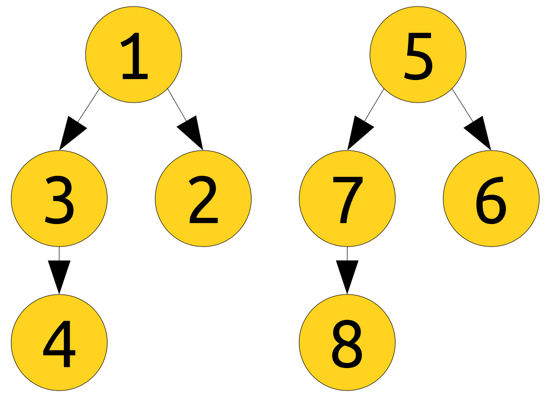
Draw what happens if we ***enqueue*** the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 into a binomial heap.



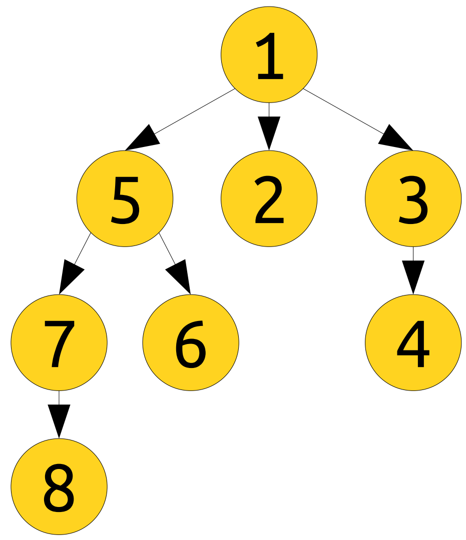
Draw what happens if we ***enqueue*** the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 into a binomial heap.



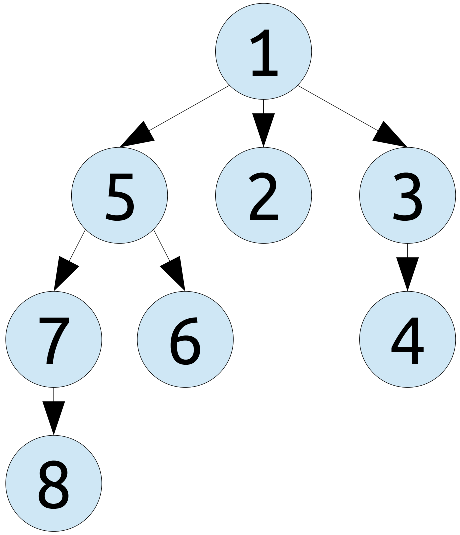
Draw what happens if we ***enqueue*** the numbers
1, 2, 3, 4, 5, 6, 7, 8, and 9 into a binomial heap.



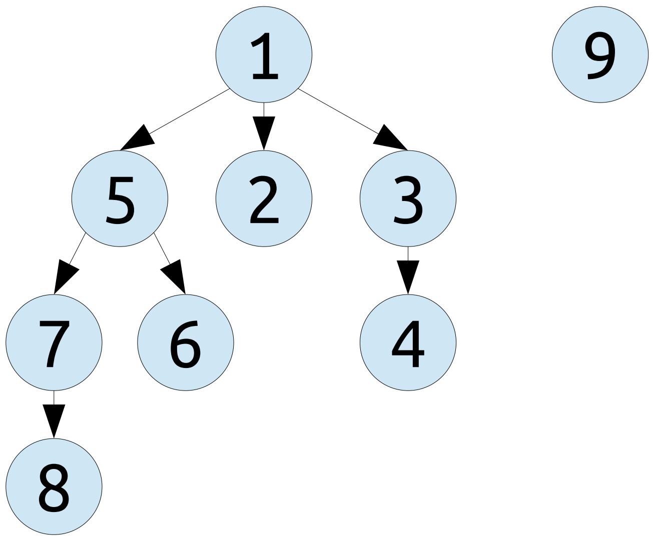
Draw what happens if we ***enqueue*** the numbers
1, 2, 3, 4, 5, 6, 7, 8, and 9 into a binomial heap.



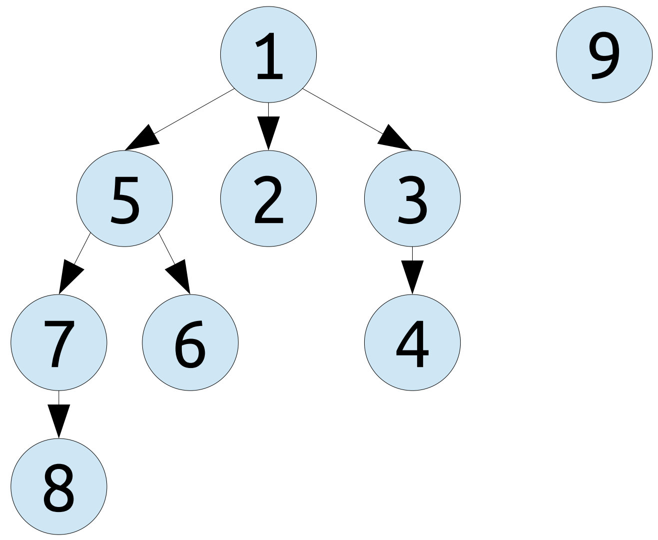
Draw what happens if we ***enqueue*** the numbers
1, 2, 3, 4, 5, 6, 7, 8, and 9 into a binomial heap.



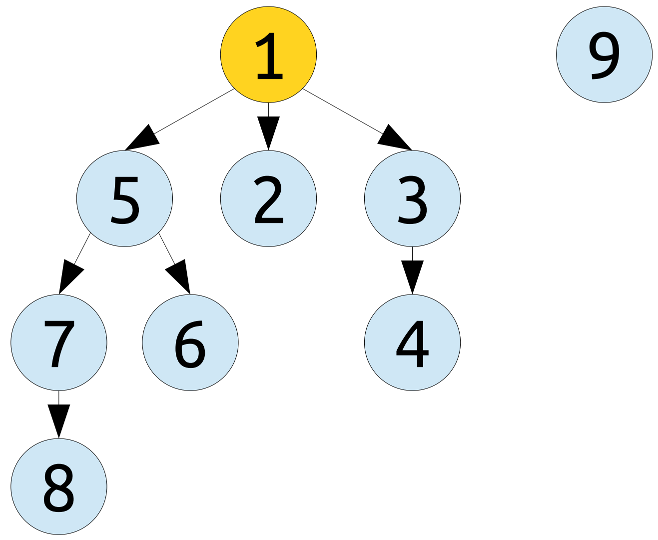
Draw what happens if we ***enqueue*** the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 into a binomial heap.



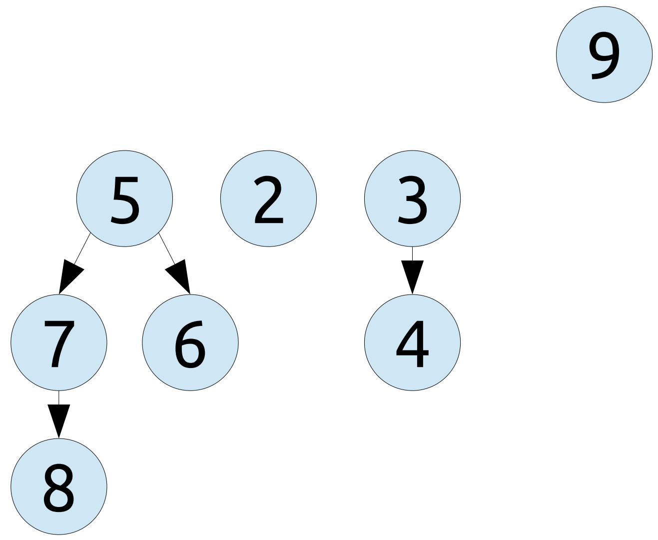
Draw what happens if we ***enqueue*** the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 into a binomial heap.



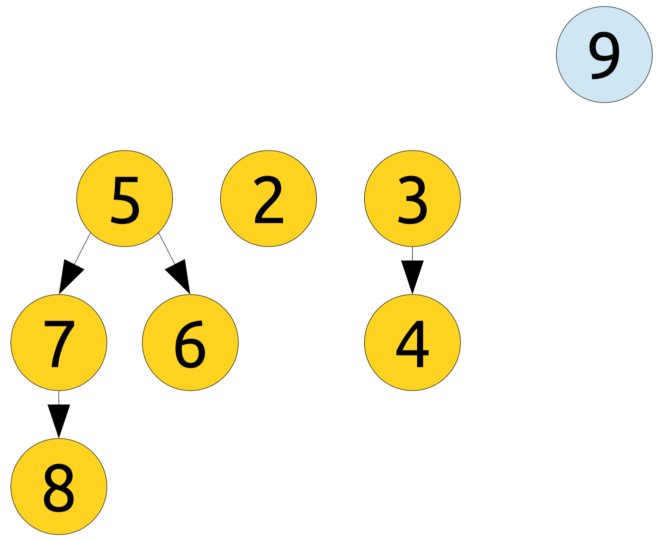
Draw what happens after performing an ***extract-min*** in this binomial heap.



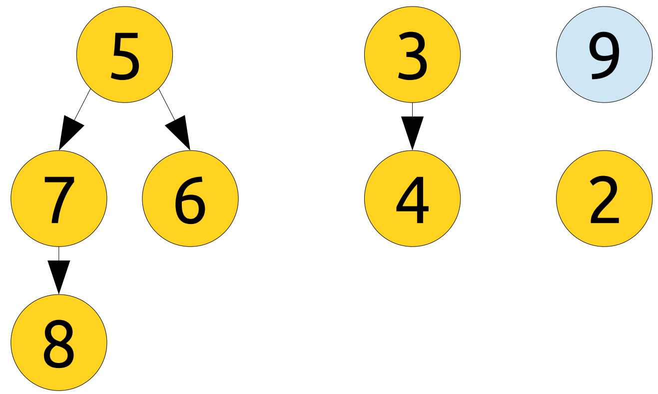
Draw what happens after performing an ***extract-min*** in this binomial heap.



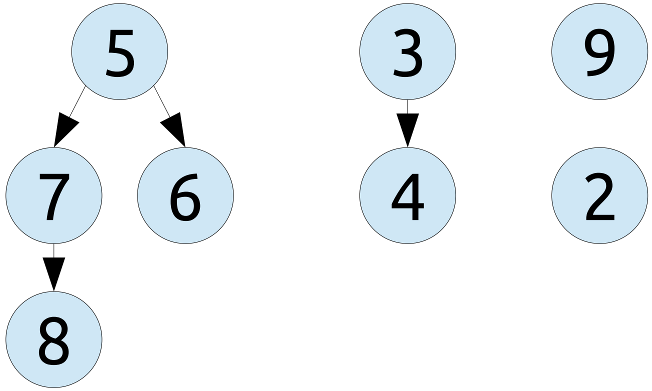
Draw what happens after performing an ***extract-min*** in this binomial heap.



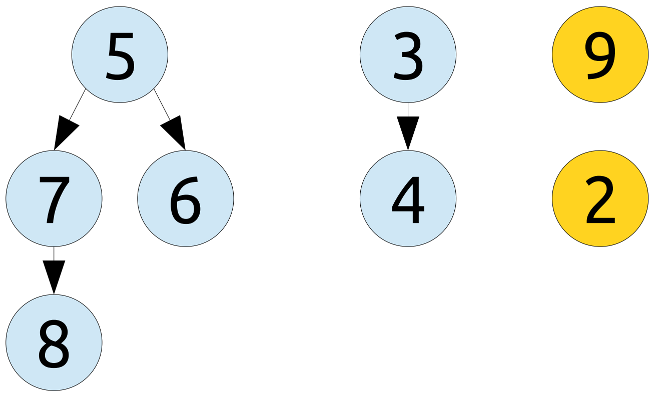
Draw what happens after performing an ***extract-min*** in this binomial heap.



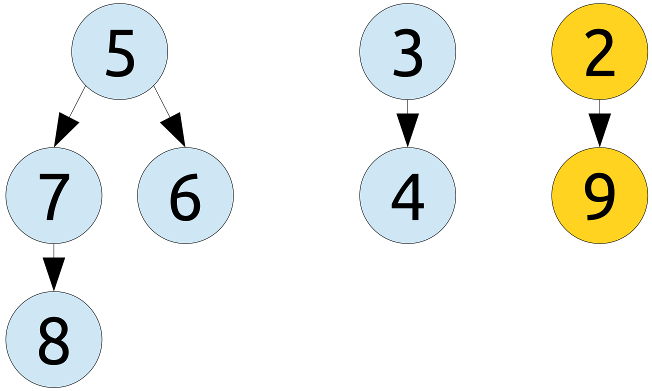
Draw what happens after performing an ***extract-min*** in this binomial heap.



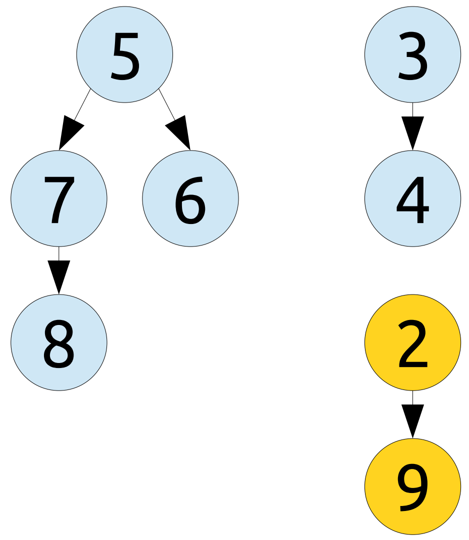
Draw what happens after performing an ***extract-min*** in this binomial heap.



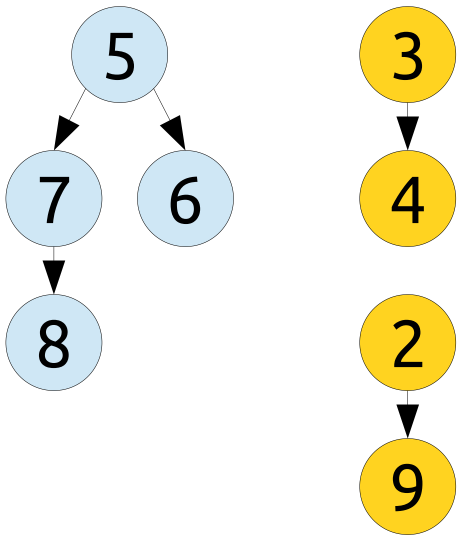
Draw what happens after performing an ***extract-min*** in this binomial heap.



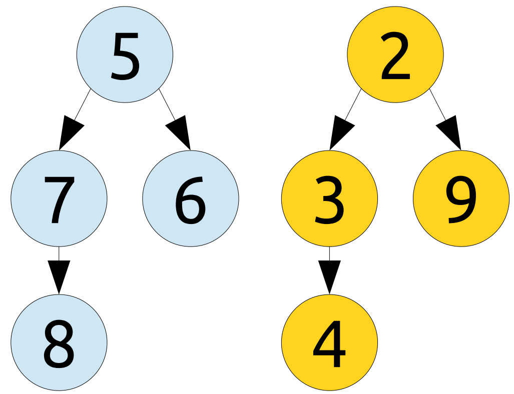
Draw what happens after performing an ***extract-min*** in this binomial heap.



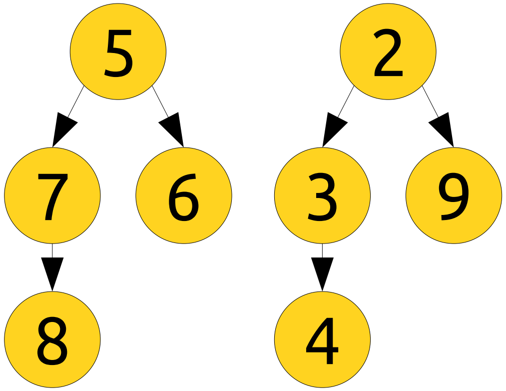
Draw what happens after performing an ***extract-min*** in this binomial heap.



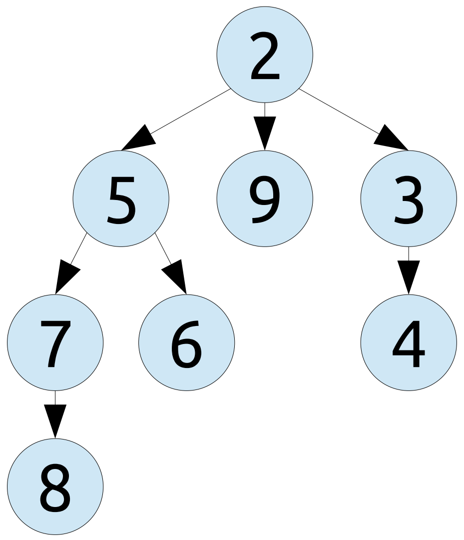
Draw what happens after performing an
extract-min in this binomial heap.



Draw what happens after performing an
extract-min in this binomial heap.



Draw what happens after performing an ***extract-min*** in this binomial heap.



Draw what happens after performing an ***extract-min*** in this binomial heap.

Where We Stand

- Here's the current scorecard for the binomial heap.
- This is a fast, elegant, and clever data structure.
- ***Question:*** Can we do better?

Binomial Heap

- ***enqueue***: O(log n)
- ***find-min***: O(log n)
- ***extract-min***: O(log n)
- ***meld***: O(log m + log n).

Time-Out for Announcements!

Problem Set Four

- Problem Set Three was due today at 1:00PM.
  - Need more time? Use a late day to extend the deadline by 24 hours, or two late days to extend it by 48 hours.
- Problem Set Four (***Balanced Trees***) goes out today. It's due on Tuesday, May 13th at 1:00PM.
  - Play around with balanced binary search trees and data structure isometries.
  - Use augmented trees to solve problems much faster than seems feasible at first glance.
- As usual, ping us if you have any questions!

Back to CS166!

Where We Stand

- ***Theorem:*** No comparison-based priority queue structure can have ***enqueue*** and ***extract-min*** each take time $o(\log n)$.

Binomial Heap

- ***enqueue***: $O(\log n)$
- ***find-min***: $O(\log n)$
- ***extract-min***: $O(\log n)$
- ***meld***: $O(\log m + \log n)$.

Why? Answer at ***https://cs166.stanford.edu/pollev***

Where We Stand

- ***Theorem:*** No comparison-based priority queue structure can have ***enqueue*** and ***extract-min*** each take time $o(\log n)$.
- ***Proof:*** Suppose these operations each take time $o(\log n)$. Then we could sort n elements by perform n ***enqueue***s and then n ***extract-min***s in time $o(n \log n)$. This is impossible with comparison-based algorithms. ■

Binomial Heap

- ***enqueue***: $O(\log n)$
- ***find-min***: $O(\log n)$
- ***extract-min***: $O(\log n)$
- ***meld***: $O(\log m + \log n)$.

Where We Stand

- We can't make both ***enqueue*** and ***extract-min*** run in time $o(\log n)$.
- However, we could conceivably make one of them faster.
- ***Question:*** Which one should we prioritize?
- Probably ***enqueue***, since we aren't guaranteed to have to remove all added items.
- ***Goal:*** Make ***enqueue*** take time O(1).

Binomial Heap

- ***enqueue***: $O(\log n)$
- ***find-min***: $O(\log n)$
- ***extract-min***: $O(\log n)$
- ***meld***: $O(\log m + \log n)$.

Where We Stand

- The ***enqueue*** operation is implemented in terms of ***meld***.
- If we want ***enqueue*** to run in time O(1), we'll need ***meld*** to take time O(1).
- How could we accomplish this?

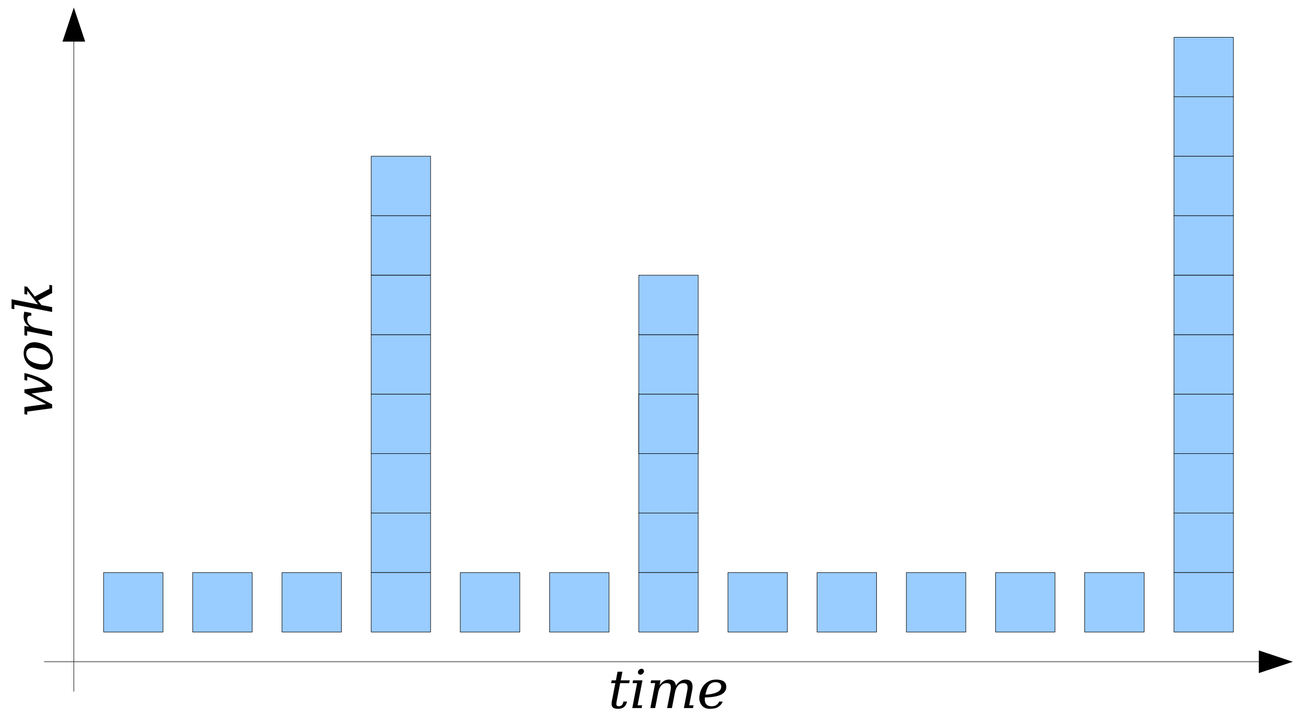
Binomial Heap

- ***enqueue***: O(log n)
- ***find-min***: O(log n)
- ***extract-min***: O(log n)
- ***meld***: O(log m + log n).

Thinking With Amortization

Refresher: Amortization

- In an amortized efficient data structure, some operations can take much longer than others, provided that previous operations didn't take too long to finish.
- Think dishwashers: you may have to do a big cleanup at some point, but that's because you did basically no work to wash all the dishes you placed in the dishwasher.

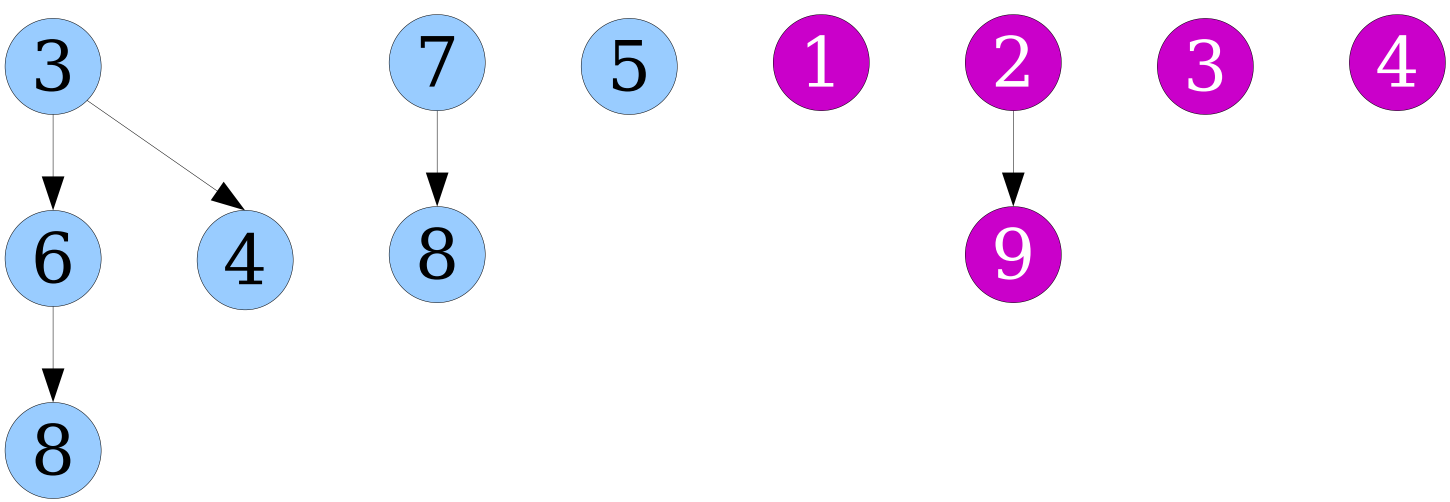


Lazy Melding

- Consider the following lazy ***meld***ing approach:

To meld together two binomial heaps, just combine the two sets of trees together.

- ***Intuition:*** Why do any work to organize keys if we're not going to do an ***extract-min***? We'll worry about cleanup then.

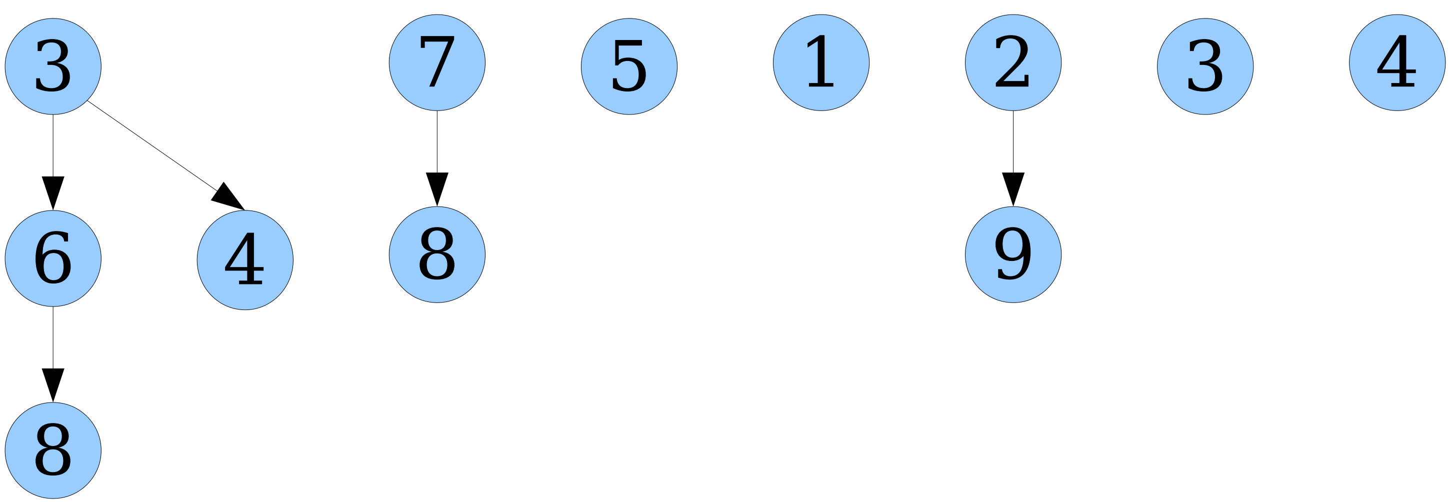


Lazy Melding

- Consider the following lazy ***meld***ing approach:

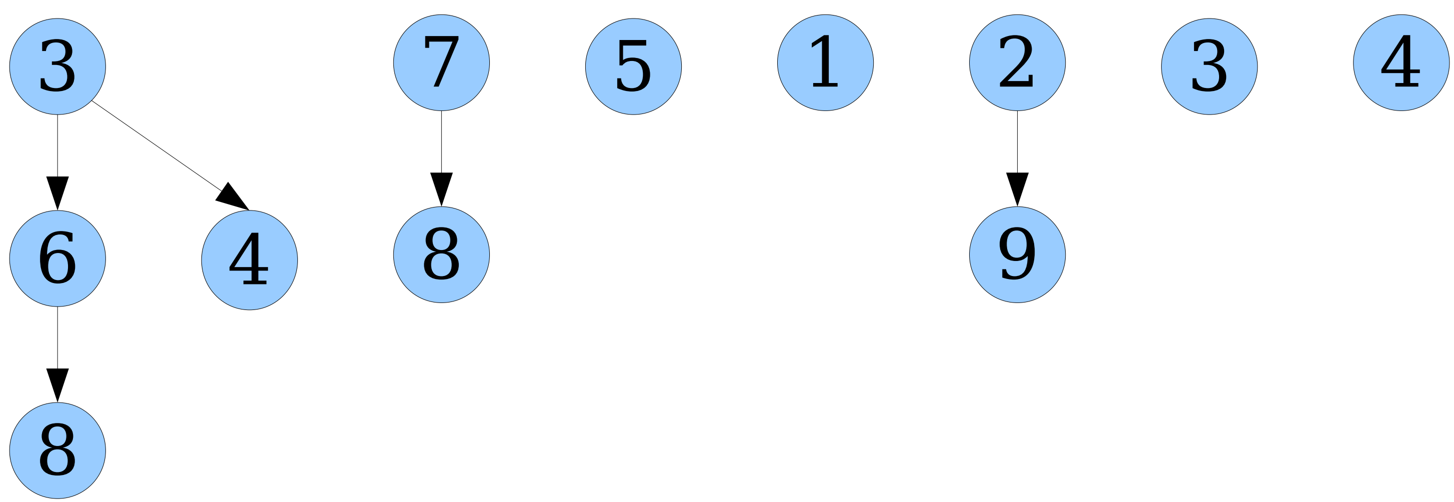
To meld together two binomial heaps, just combine the two sets of trees together.

- ***Intuition:*** Why do any work to organize keys if we're not going to do an ***extract-min***? We'll worry about cleanup then.



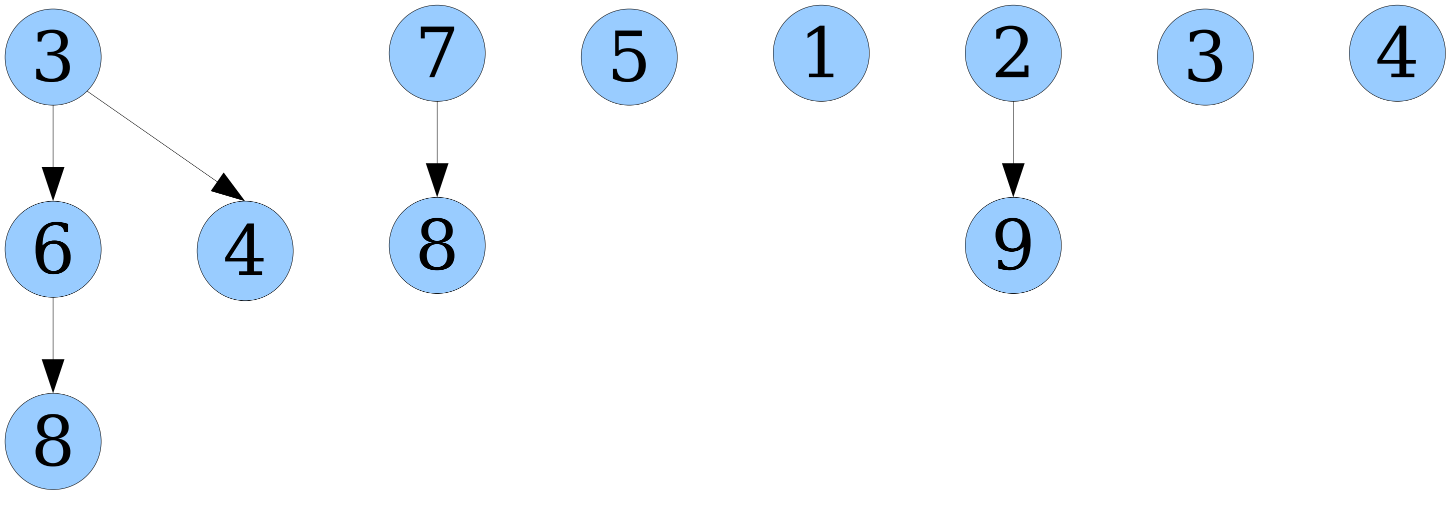
Lazy Melding

- If we store our list of trees as circularly, doubly-linked lists, we can concatenate tree lists in time O(1).
  - Cost of a ***meld***: **O(1)**.
  - Cost of an ***enqueue***: **O(1)**.
- If it sounds too good to be true, it probably is.



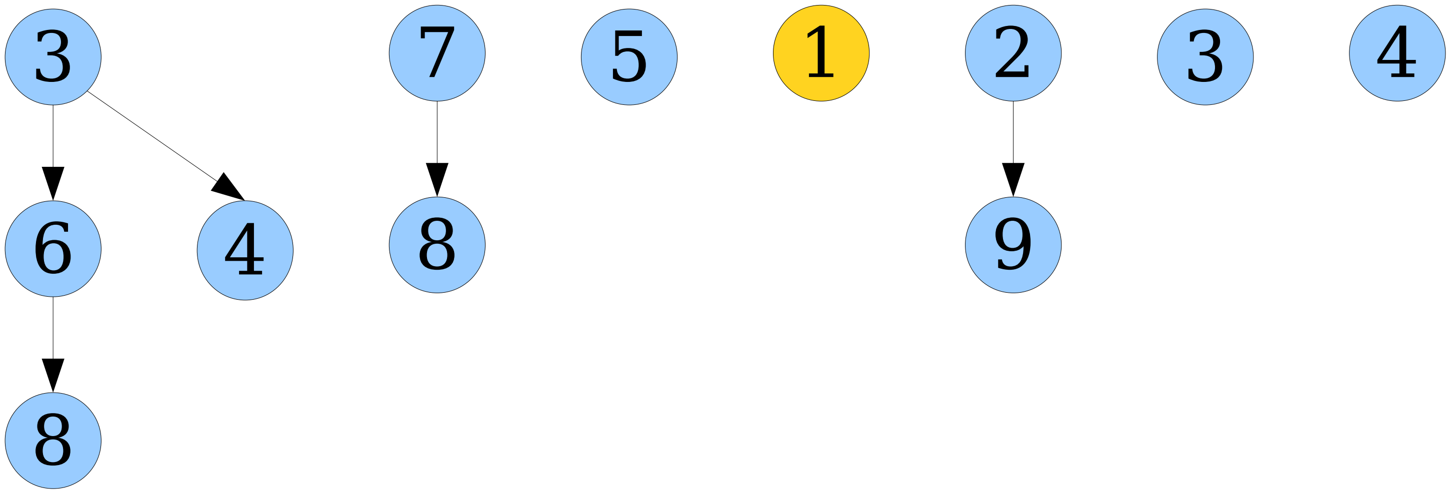
Lazy Melding

- Imagine that we implement ***extract-min*** the same way as before:
  - Find the packet with the minimum.
  - “Fracture” that packet to expose smaller packets.
  - Meld those packets back in with the master list.
- What happens if we do this with lazy melding?



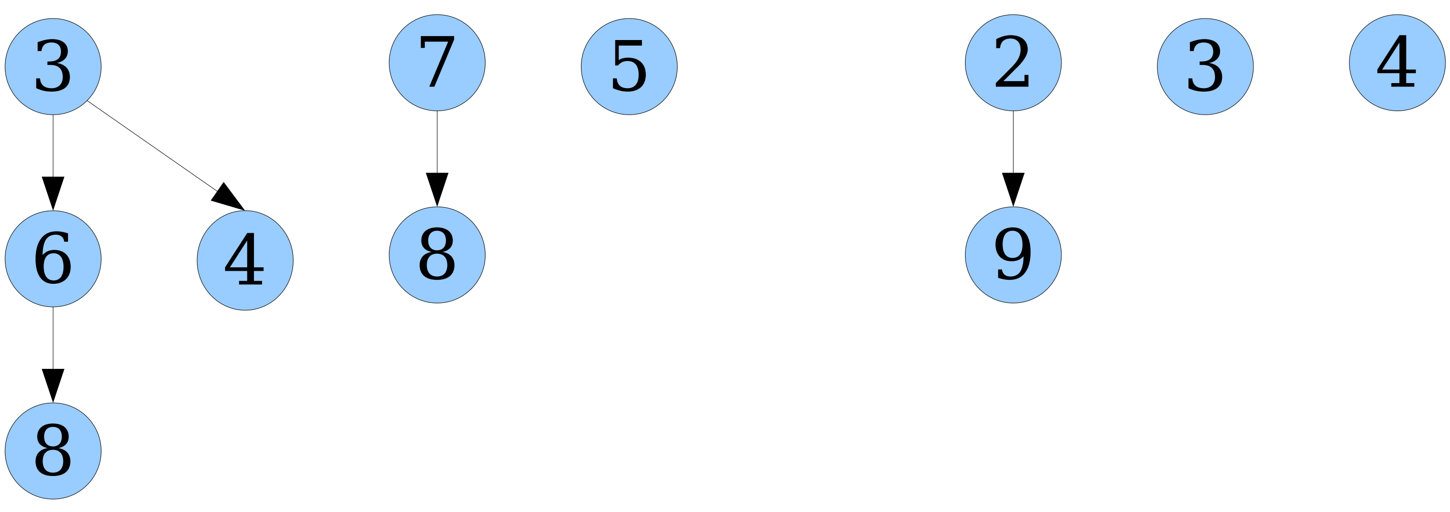
Lazy Melding

- Imagine that we implement ***extract-min*** the same way as before:
  - Find the packet with the minimum.
  - “Fracture” that packet to expose smaller packets.
  - Meld those packets back in with the master list.
- What happens if we do this with lazy melding?



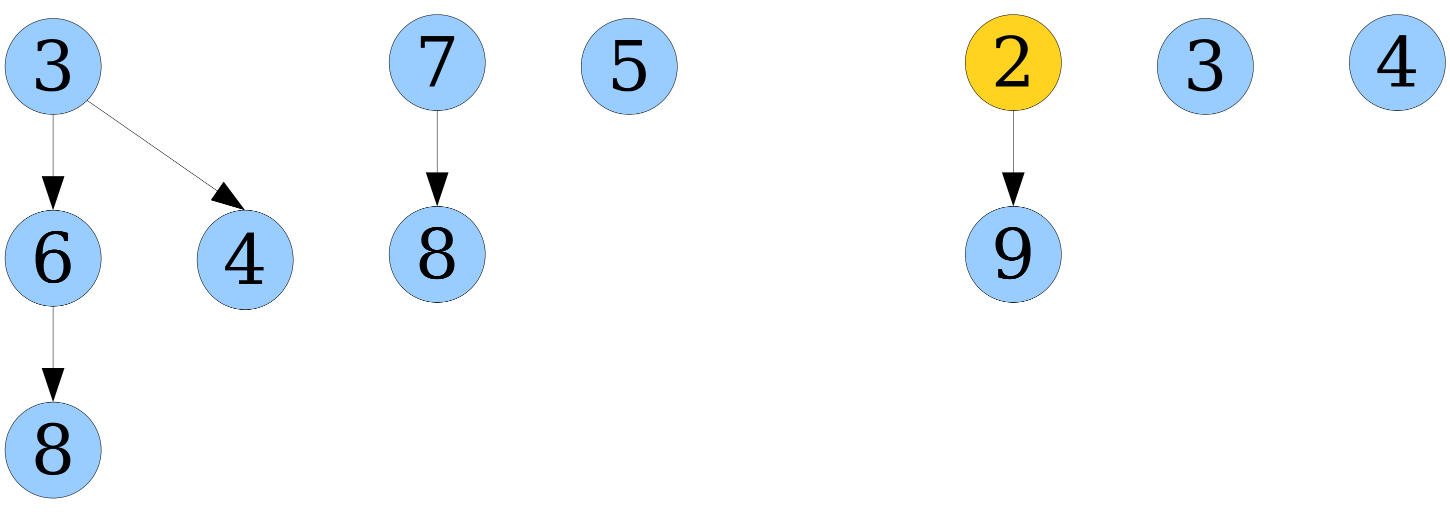
Lazy Melding

- Imagine that we implement ***extract-min*** the same way as before:
  - Find the packet with the minimum.
  - “Fracture” that packet to expose smaller packets.
  - Meld those packets back in with the master list.
- What happens if we do this with lazy melding?



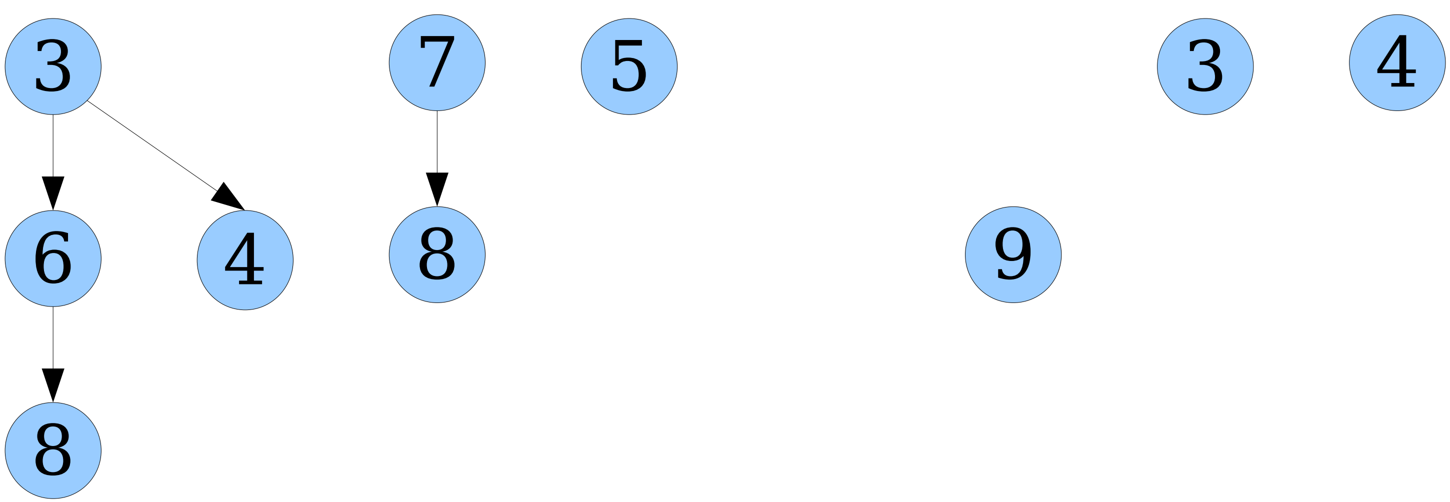
Lazy Melding

- Imagine that we implement ***extract-min*** the same way as before:
  - Find the packet with the minimum.
  - "Fracture" that packet to expose smaller packets.
  - Meld those packets back in with the master list.
- What happens if we do this with lazy melding?



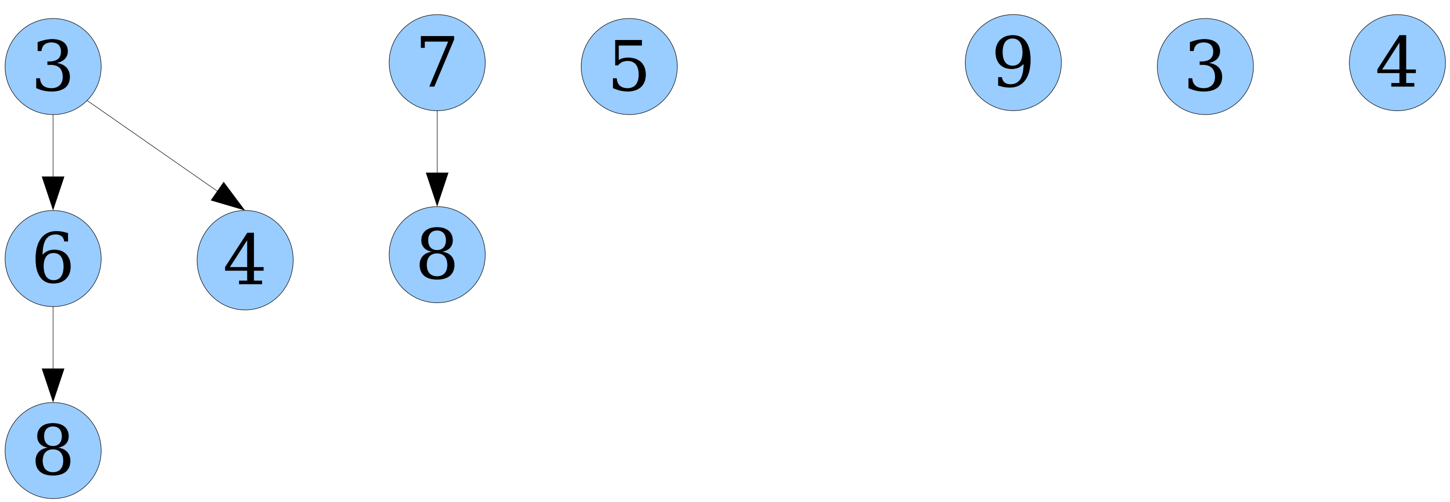
Lazy Melding

- Imagine that we implement ***extract-min*** the same way as before:
  - Find the packet with the minimum.
  - “Fracture” that packet to expose smaller packets.
  - Meld those packets back in with the master list.
- What happens if we do this with lazy melding?



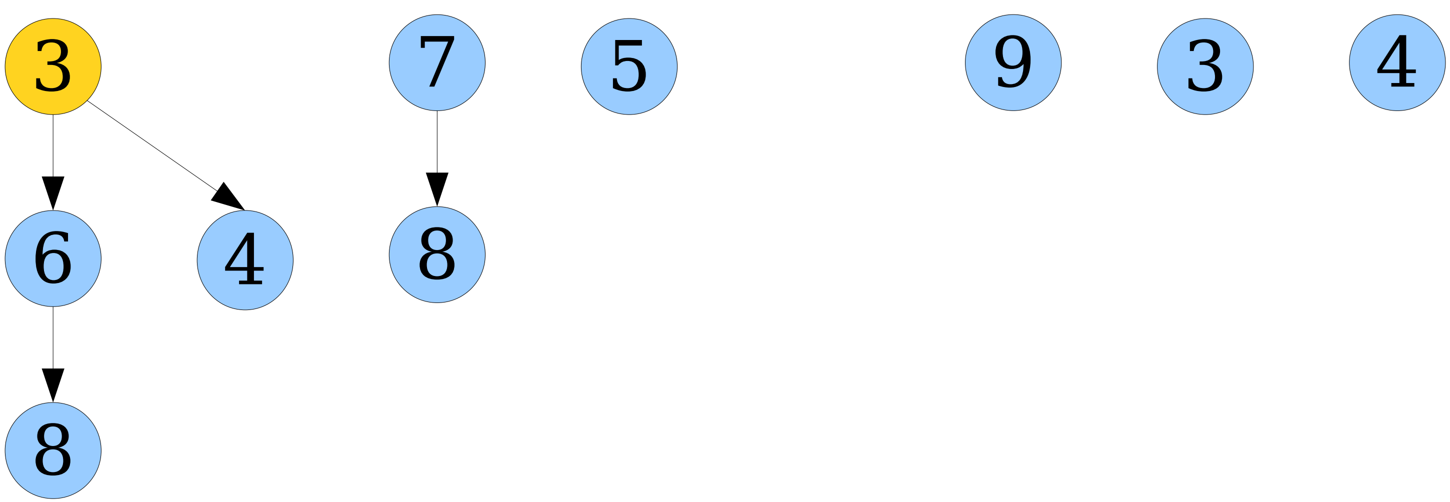
Lazy Melding

- Imagine that we implement ***extract-min*** the same way as before:
  - Find the packet with the minimum.
  - “Fracture” that packet to expose smaller packets.
  - Meld those packets back in with the master list.
- What happens if we do this with lazy melding?



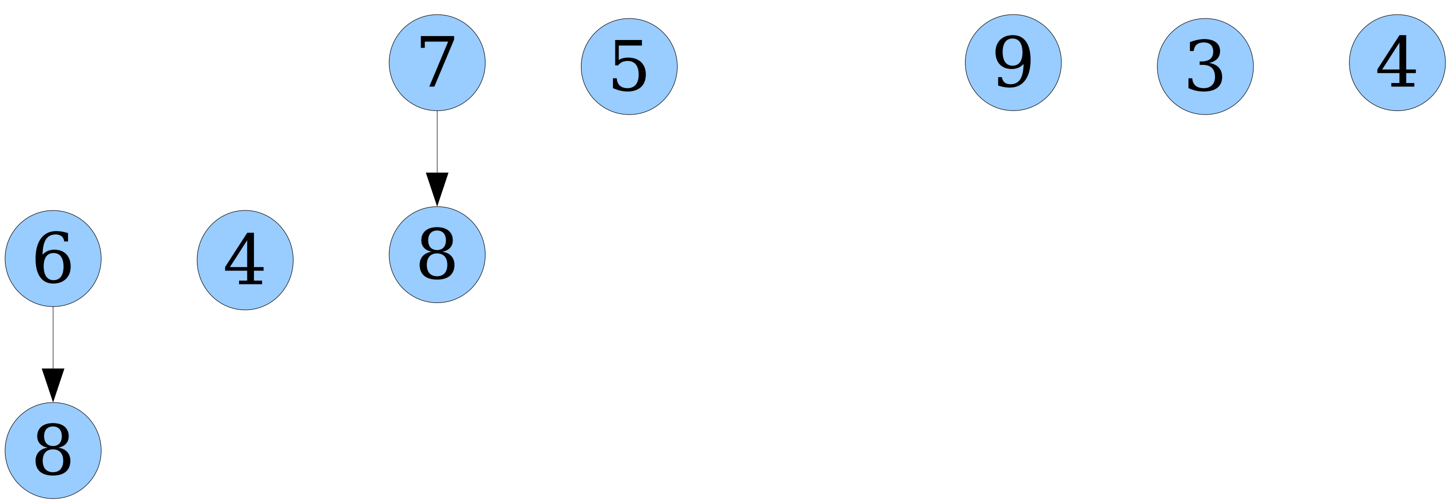
Lazy Melding

- Imagine that we implement ***extract-min*** the same way as before:
  - Find the packet with the minimum.
  - “Fracture” that packet to expose smaller packets.
  - Meld those packets back in with the master list.
- What happens if we do this with lazy melding?



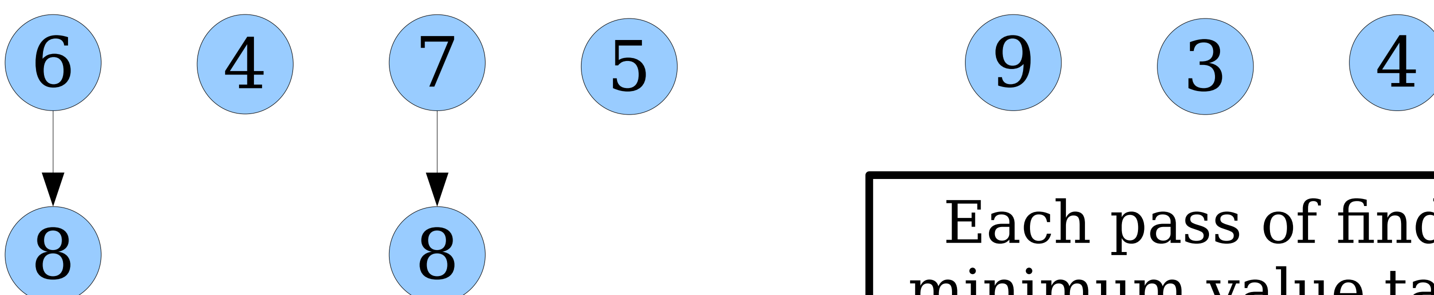
Lazy Melding

- Imagine that we implement ***extract-min*** the same way as before:
  - Find the packet with the minimum.
  - "Fracture" that packet to expose smaller packets.
  - Meld those packets back in with the master list.
- What happens if we do this with lazy melding?



Lazy Melding

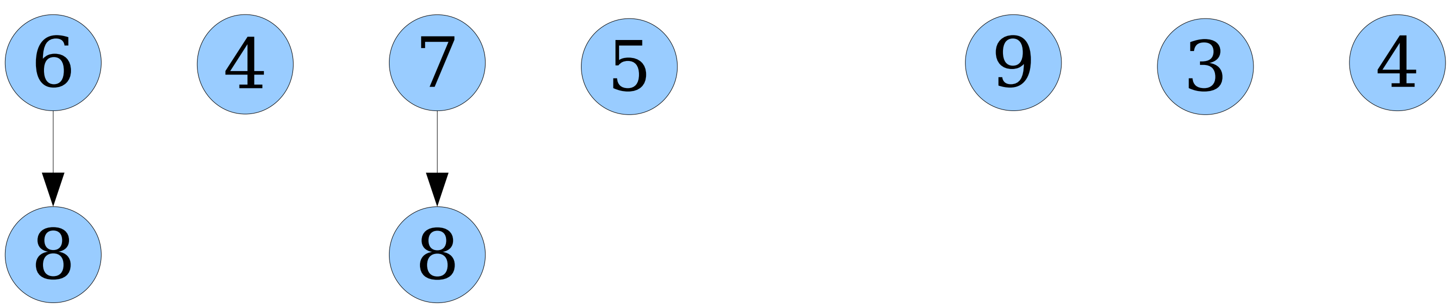
- Imagine that we implement ***extract-min*** the same way as before:
  - Find the packet with the minimum.
  - “Fracture” that packet to expose smaller packets.
  - Meld those packets back in with the master list.
- What happens if we do this with lazy melding?



Each pass of finding the minimum value takes time $\Theta(n)$ in the worst case. We’ve lost our nice runtime guarantees!

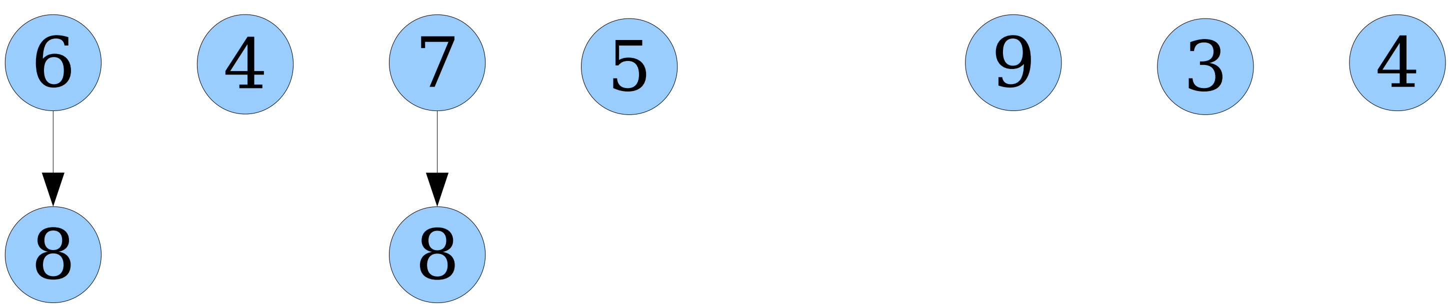
Washing the Dishes

- Every ***meld*** (and ***enqueue***) creates some “dirty dishes” (small trees) that we need to clean up later.
- If we never clean them up, then our ***extract-min*** will be too slow to be usable.
- ***Idea:*** Change ***extract-min*** to “wash the dishes” and make things look nice and pretty again.
- ***Question:*** What does “wash the dishes” mean here?



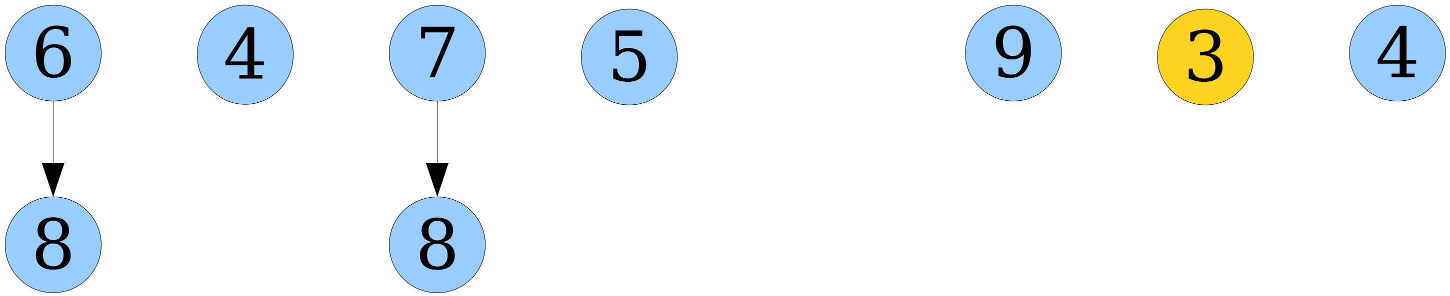
Washing the Dishes

- With our eager ***meld*** (and ***enqueue***) strategy, our priority queue never had more than one tree of each order.
- This kept the number of trees low, which is why each operation was so fast.
- ***Idea:*** After doing an ***extract-min***, do a ***coalesce step*** to ensure there's at most one tree of each order. This gets us to where we would be if we had been doing cleanup as we go.



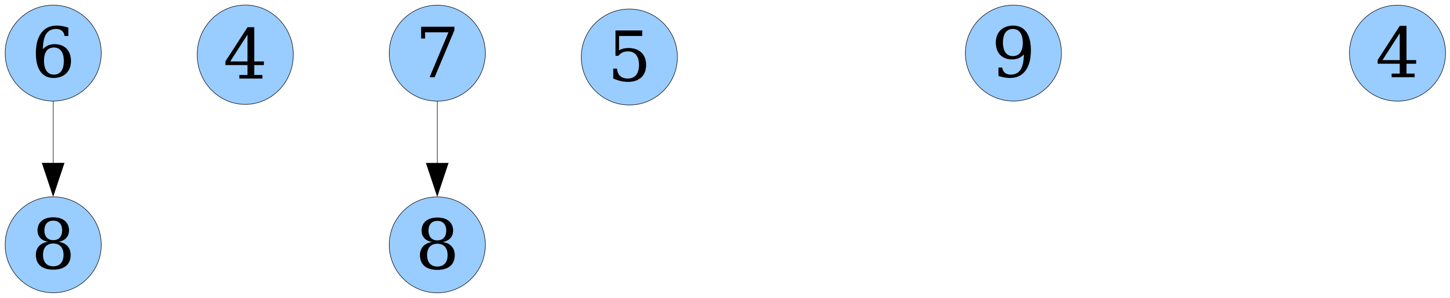
Washing the Dishes

- With our eager ***meld*** (and ***enqueue***) strategy, our priority queue never had more than one tree of each order.
- This kept the number of trees low, which is why each operation was so fast.
- ***Idea:*** After doing an ***extract-min***, do a ***coalesce step*** to ensure there's at most one tree of each order. This gets us to where we would be if we had been doing cleanup as we go.



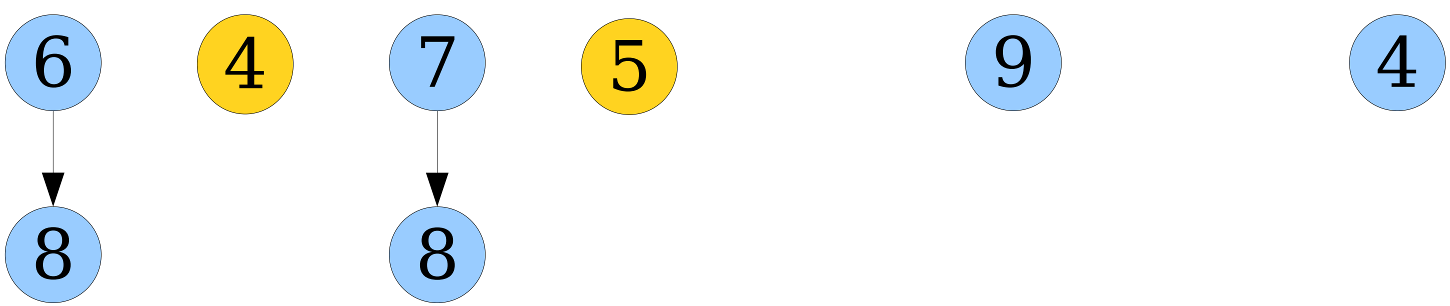
Washing the Dishes

- With our eager ***meld*** (and ***enqueue***) strategy, our priority queue never had more than one tree of each order.
- This kept the number of trees low, which is why each operation was so fast.
- ***Idea:*** After doing an ***extract-min***, do a ***coalesce step*** to ensure there’s at most one tree of each order. This gets us to where we would be if we had been doing cleanup as we go.



Washing the Dishes

- With our eager ***meld*** (and ***enqueue***) strategy, our priority queue never had more than one tree of each order.
- This kept the number of trees low, which is why each operation was so fast.
- ***Idea:*** After doing an ***extract-min***, do a ***coalesce step*** to ensure there's at most one tree of each order. This gets us to where we would be if we had been doing cleanup as we go.



Washing the Dishes

- With our eager ***meld*** (and ***enqueue***) strategy, our priority queue never had more than one tree of each order.
- This kept the number of trees low, which is why each operation was so fast.
- ***Idea:*** After doing an ***extract-min***, do a ***coalesce step*** to ensure there's at most one tree of each order. This gets us to where we would be if we had been doing cleanup as we go.

Washing the Dishes

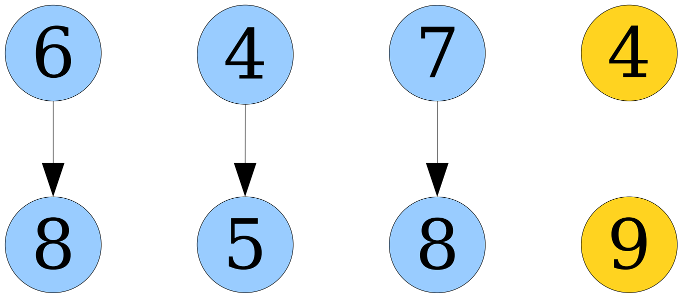
- With our eager ***meld*** (and ***enqueue***) strategy, our priority queue never had more than one tree of each order.
- This kept the number of trees low, which is why each operation was so fast.
- ***Idea:*** After doing an ***extract-min***, do a ***coalesce step*** to ensure there's at most one tree of each order. This gets us to where we would be if we had been doing cleanup as we go.

Washing the Dishes

- With our eager ***meld*** (and ***enqueue***) strategy, our priority queue never had more than one tree of each order.
- This kept the number of trees low, which is why each operation was so fast.
- ***Idea:*** After doing an ***extract-min***, do a ***coalesce step*** to ensure there's at most one tree of each order. This gets us to where we would be if we had been doing cleanup as we go.

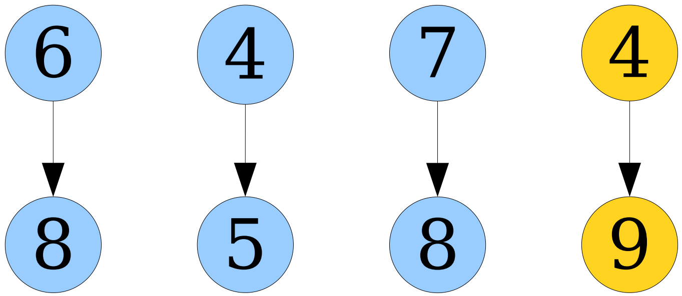
Washing the Dishes

- With our eager ***meld*** (and ***enqueue***) strategy, our priority queue never had more than one tree of each order.
- This kept the number of trees low, which is why each operation was so fast.
- ***Idea:*** After doing an ***extract-min***, do a ***coalesce step*** to ensure there's at most one tree of each order. This gets us to where we would be if we had been doing cleanup as we go.



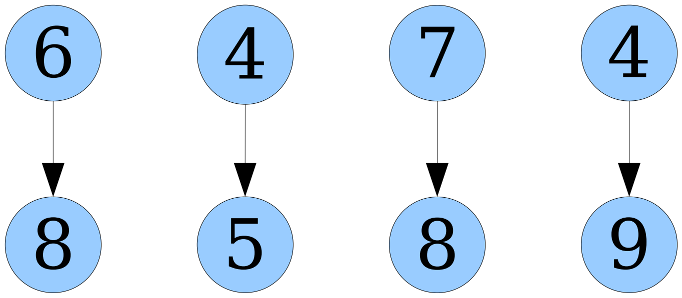
Washing the Dishes

- With our eager ***meld*** (and ***enqueue***) strategy, our priority queue never had more than one tree of each order.
- This kept the number of trees low, which is why each operation was so fast.
- ***Idea:*** After doing an ***extract-min***, do a ***coalesce step*** to ensure there's at most one tree of each order. This gets us to where we would be if we had been doing cleanup as we go.



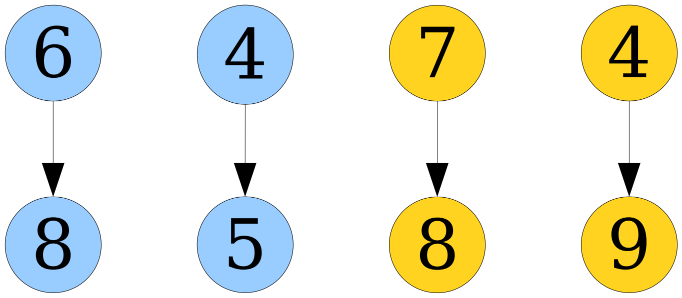
Washing the Dishes

- With our eager ***meld*** (and ***enqueue***) strategy, our priority queue never had more than one tree of each order.
- This kept the number of trees low, which is why each operation was so fast.
- ***Idea:*** After doing an ***extract-min***, do a ***coalesce step*** to ensure there's at most one tree of each order. This gets us to where we would be if we had been doing cleanup as we go.



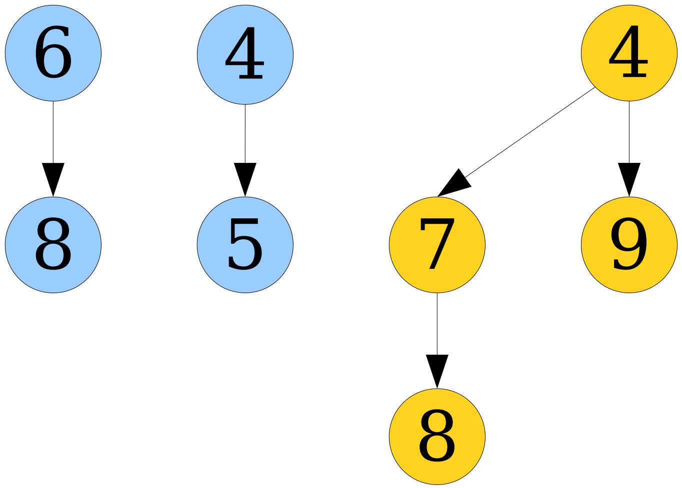
Washing the Dishes

- With our eager ***meld*** (and ***enqueue***) strategy, our priority queue never had more than one tree of each order.
- This kept the number of trees low, which is why each operation was so fast.
- ***Idea:*** After doing an ***extract-min***, do a ***coalesce step*** to ensure there's at most one tree of each order. This gets us to where we would be if we had been doing cleanup as we go.



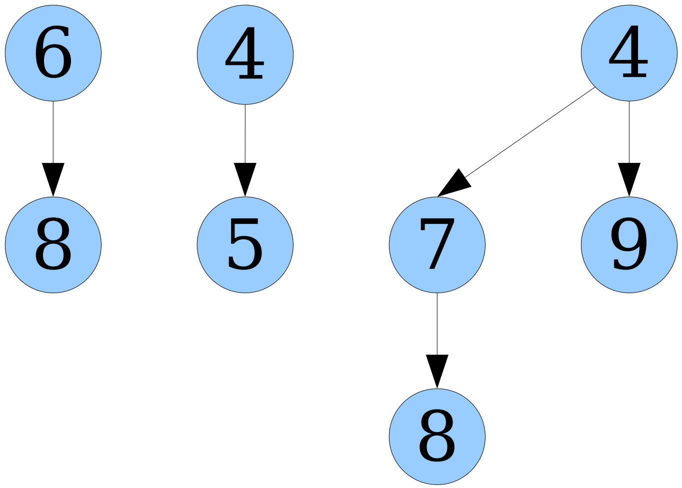
Washing the Dishes

- With our eager ***meld*** (and ***enqueue***) strategy, our priority queue never had more than one tree of each order.
- This kept the number of trees low, which is why each operation was so fast.
- ***Idea:*** After doing an ***extract-min***, do a ***coalesce step*** to ensure there's at most one tree of each order. This gets us to where we would be if we had been doing cleanup as we go.



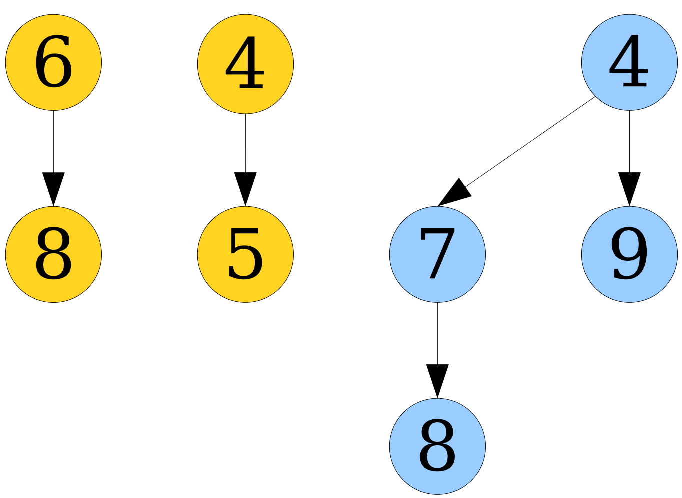
Washing the Dishes

- With our eager ***meld*** (and ***enqueue***) strategy, our priority queue never had more than one tree of each order.
- This kept the number of trees low, which is why each operation was so fast.
- ***Idea:*** After doing an ***extract-min***, do a ***coalesce step*** to ensure there's at most one tree of each order. This gets us to where we would be if we had been doing cleanup as we go.



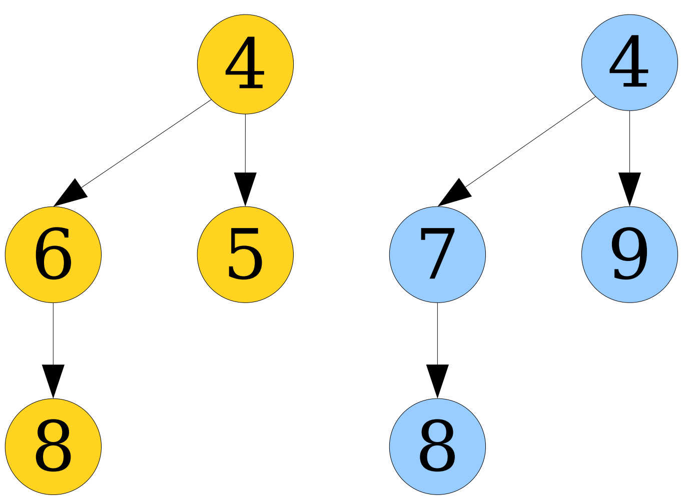
Washing the Dishes

- With our eager ***meld*** (and ***enqueue***) strategy, our priority queue never had more than one tree of each order.
- This kept the number of trees low, which is why each operation was so fast.
- ***Idea:*** After doing an ***extract-min***, do a ***coalesce step*** to ensure there’s at most one tree of each order. This gets us to where we would be if we had been doing cleanup as we go.



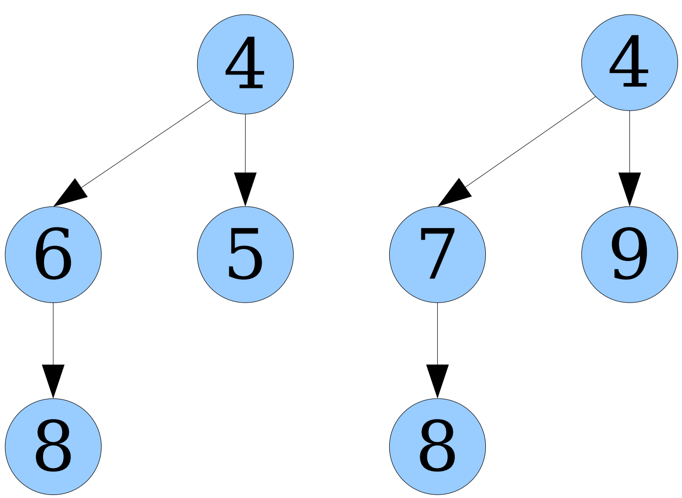
Washing the Dishes

- With our eager ***meld*** (and ***enqueue***) strategy, our priority queue never had more than one tree of each order.
- This kept the number of trees low, which is why each operation was so fast.
- ***Idea:*** After doing an ***extract-min***, do a ***coalesce step*** to ensure there's at most one tree of each order. This gets us to where we would be if we had been doing cleanup as we go.



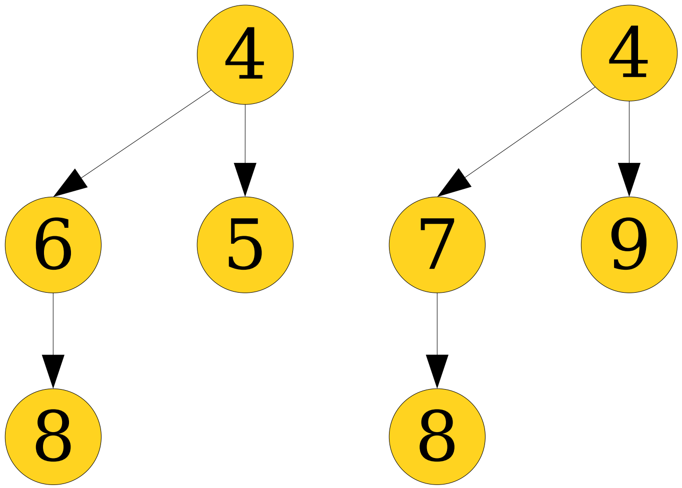
Washing the Dishes

- With our eager ***meld*** (and ***enqueue***) strategy, our priority queue never had more than one tree of each order.
- This kept the number of trees low, which is why each operation was so fast.
- ***Idea:*** After doing an ***extract-min***, do a ***coalesce step*** to ensure there's at most one tree of each order. This gets us to where we would be if we had been doing cleanup as we go.



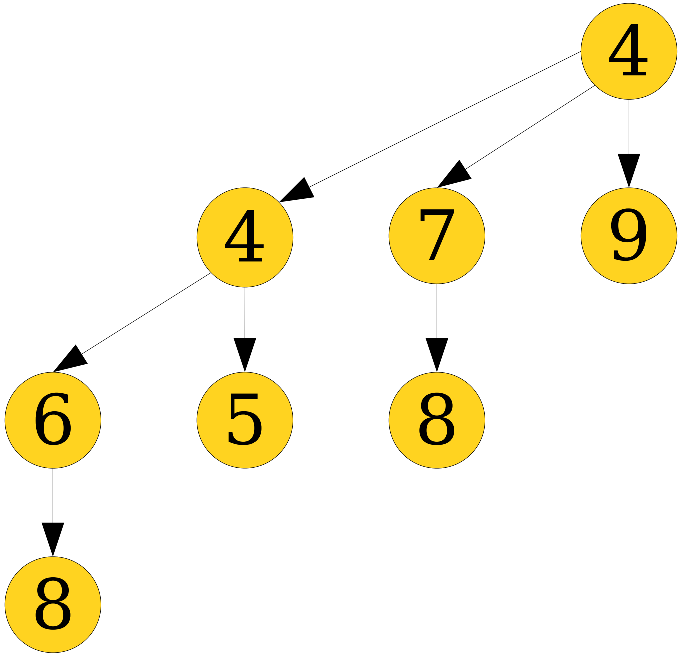
Washing the Dishes

- With our eager ***meld*** (and ***enqueue***) strategy, our priority queue never had more than one tree of each order.
- This kept the number of trees low, which is why each operation was so fast.
- ***Idea:*** After doing an ***extract-min***, do a ***coalesce step*** to ensure there's at most one tree of each order. This gets us to where we would be if we had been doing cleanup as we go.



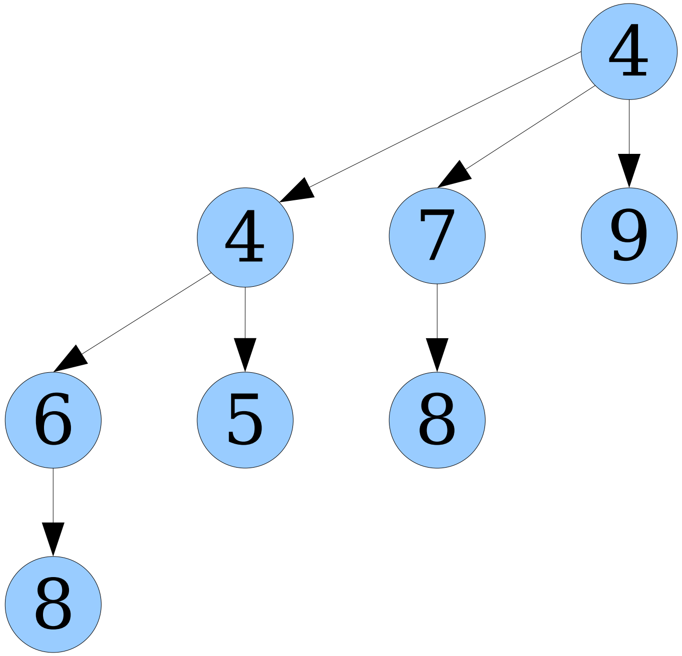
Washing the Dishes

- With our eager ***meld*** (and ***enqueue***) strategy, our priority queue never had more than one tree of each order.
- This kept the number of trees low, which is why each operation was so fast.
- ***Idea:*** After doing an ***extract-min***, do a ***coalesce step*** to ensure there’s at most one tree of each order. This gets us to where we would be if we had been doing cleanup as we go.



Washing the Dishes

- With our eager ***meld*** (and ***enqueue***) strategy, our priority queue never had more than one tree of each order.
- This kept the number of trees low, which is why each operation was so fast.
- ***Idea:*** After doing an ***extract-min***, do a ***coalesce step*** to ensure there's at most one tree of each order. This gets us to where we would be if we had been doing cleanup as we go.



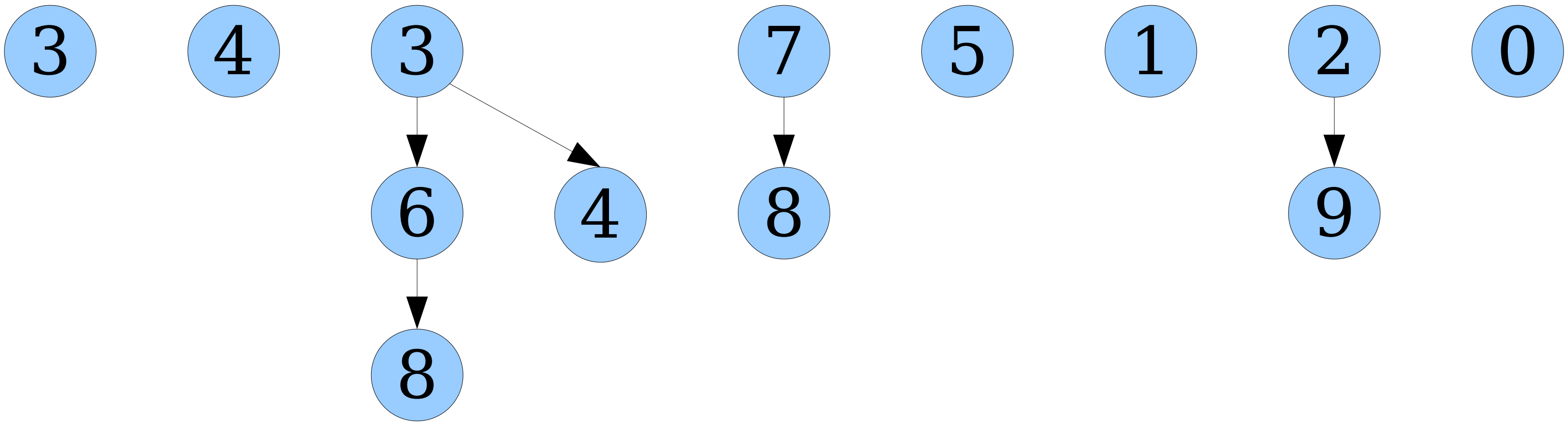
At this point, the mess is cleaned up, and we're left with what we would have had if we had been cleaning up as we go.

Where We're Going

- A ***lazy binomial heap*** is a binomial heap, modified as follows:
  - The ***meld*** operation is lazy. It just combines the two groups of trees together.
  - After doing an ***extract-min***, we do a ***coalesce*** to combine together trees until there's at most one tree of each order.
- Intuitively, we'd expect this to amortize away nicely, since the "mess" left by ***meld*** gets cleaned up later on by a future ***extract-min***.
- Questions left to answer:
  - How do we efficiently implement the ***coalesce*** operation?
  - How efficient is this approach, in an amortized sense?

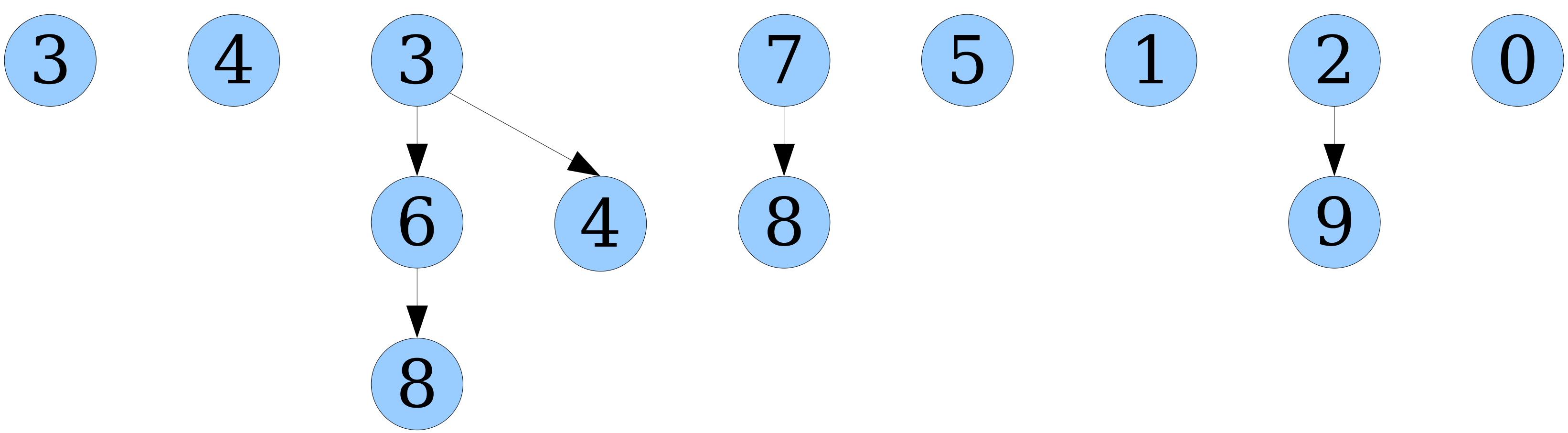
Coalescing Trees

- The ***coalesce*** step repeatedly combines trees together until there's at most one tree of each order.
- How do we implement this so that it runs quickly?



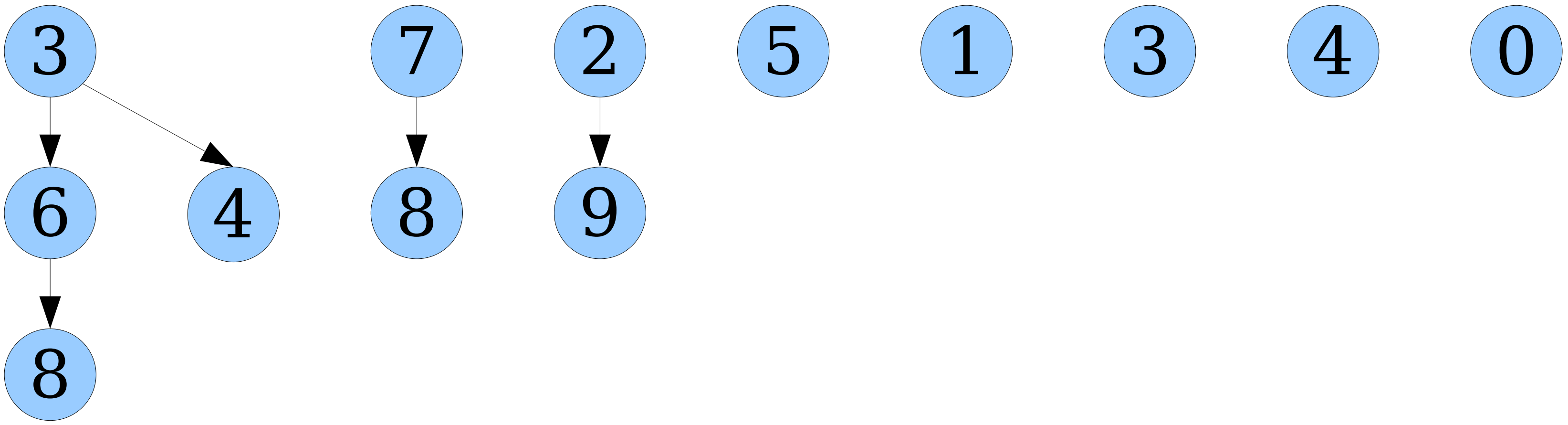
Coalescing Trees

- ***Observation:*** This would be a *lot* easier to do if all the trees were sorted by size.



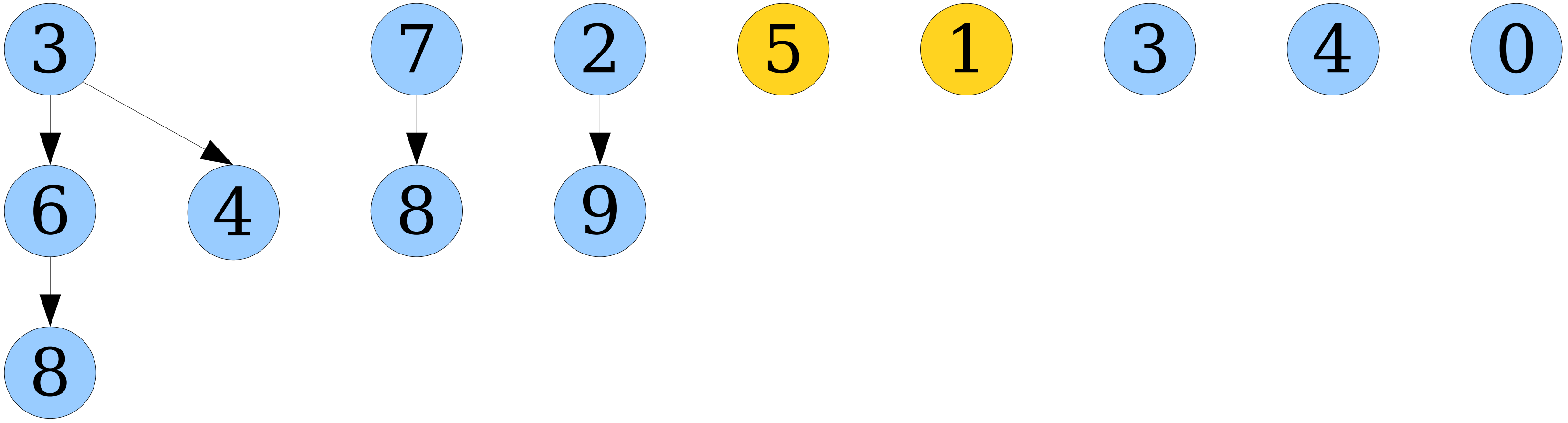
Coalescing Trees

- ***Observation:*** This would be a *lot* easier to do if all the trees were sorted by size.



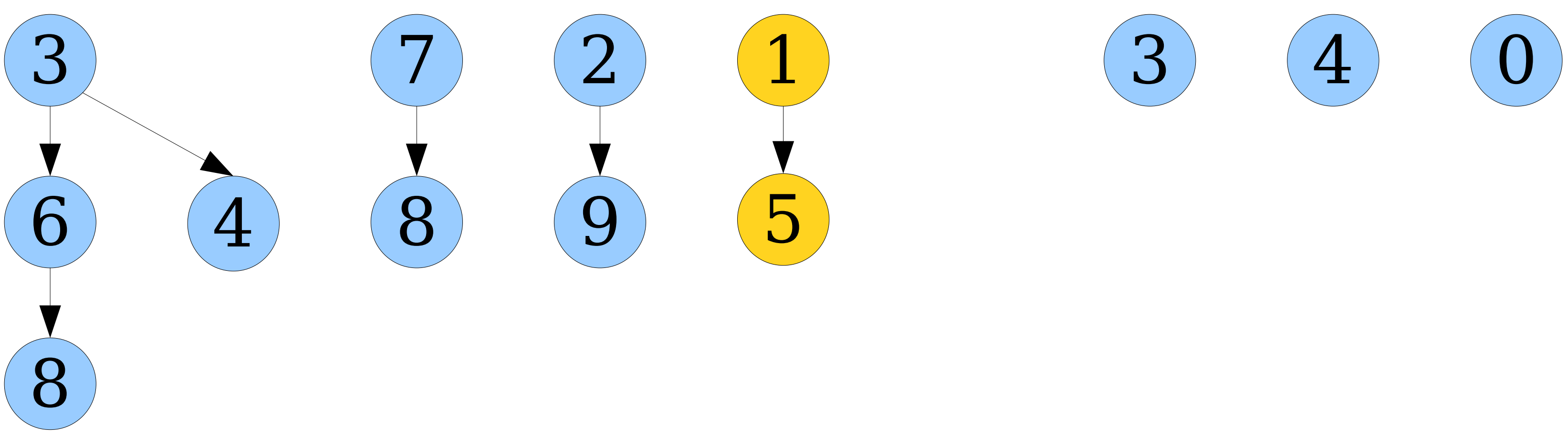
Coalescing Trees

- ***Observation:*** This would be a *lot* easier to do if all the trees were sorted by size.



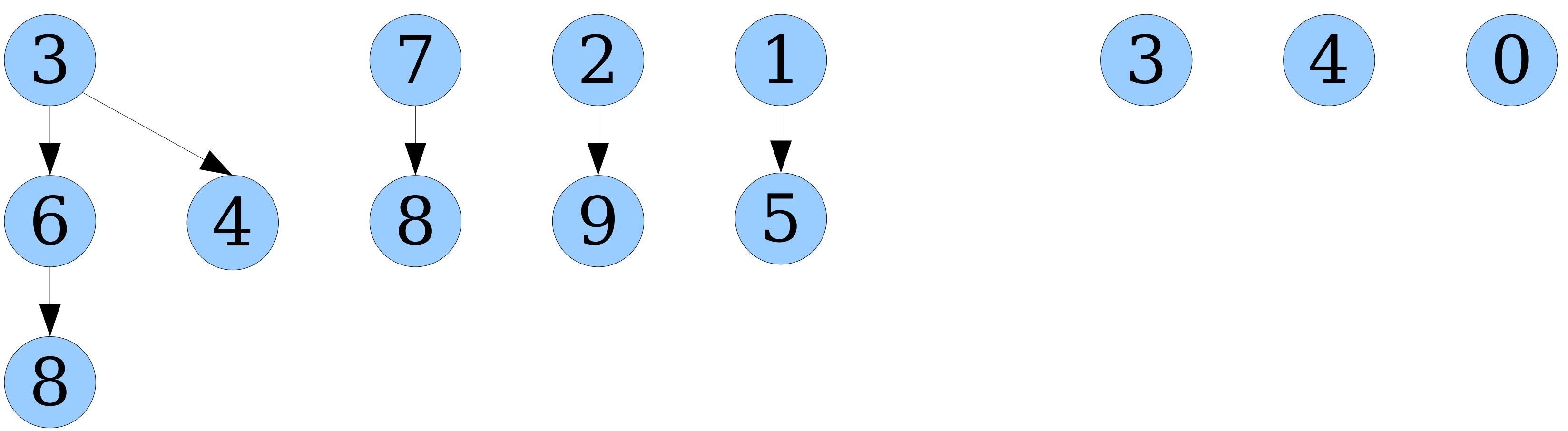
Coalescing Trees

- ***Observation:*** This would be a *lot* easier to do if all the trees were sorted by size.



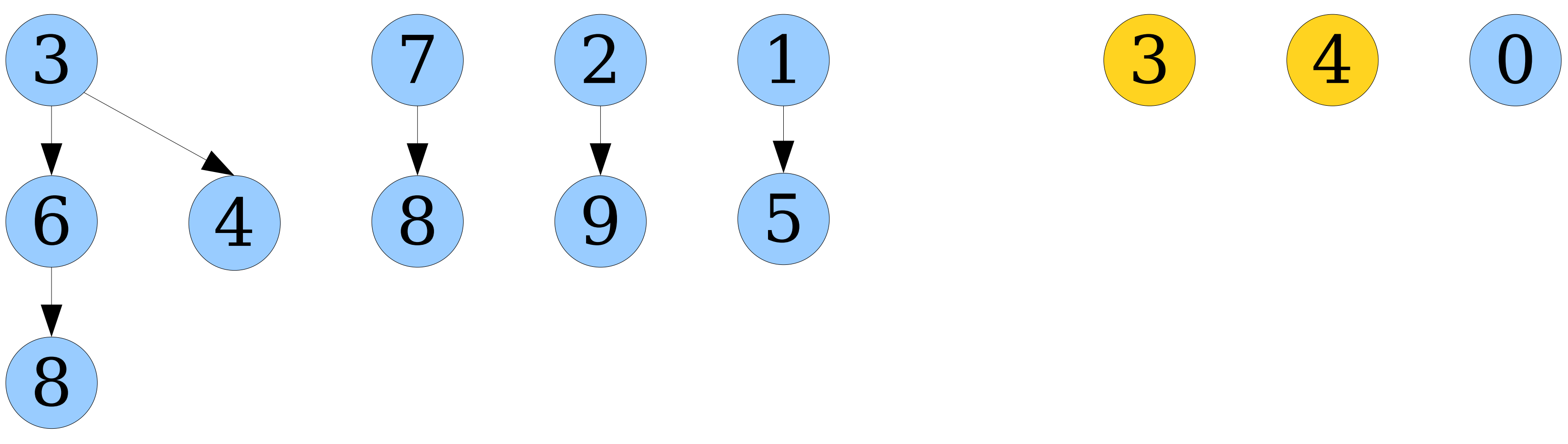
Coalescing Trees

- ***Observation:*** This would be a *lot* easier to do if all the trees were sorted by size.



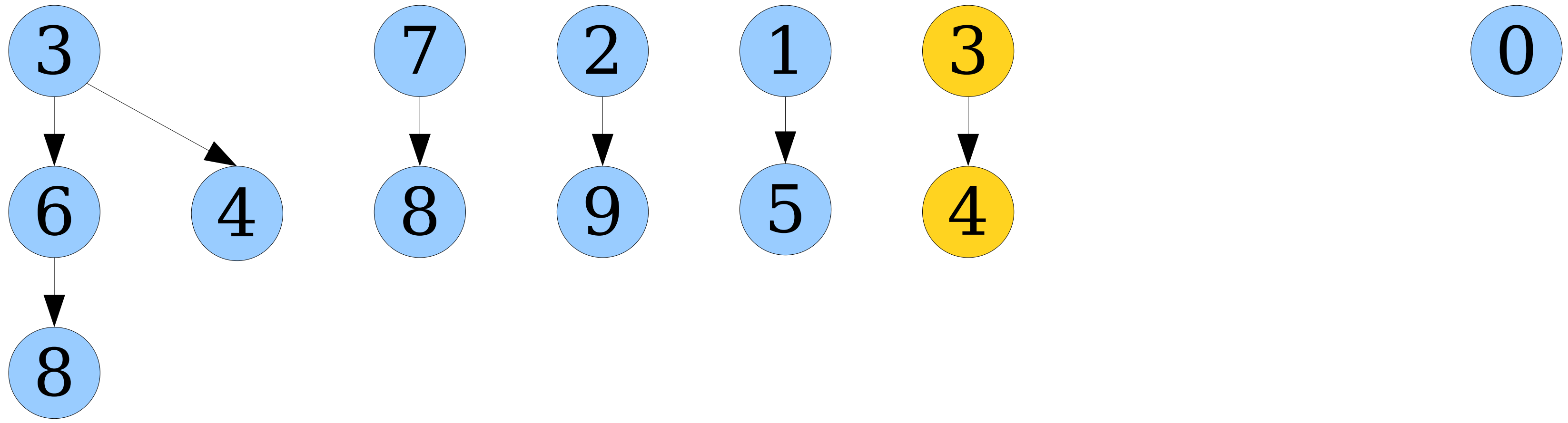
Coalescing Trees

- ***Observation:*** This would be a *lot* easier to do if all the trees were sorted by size.



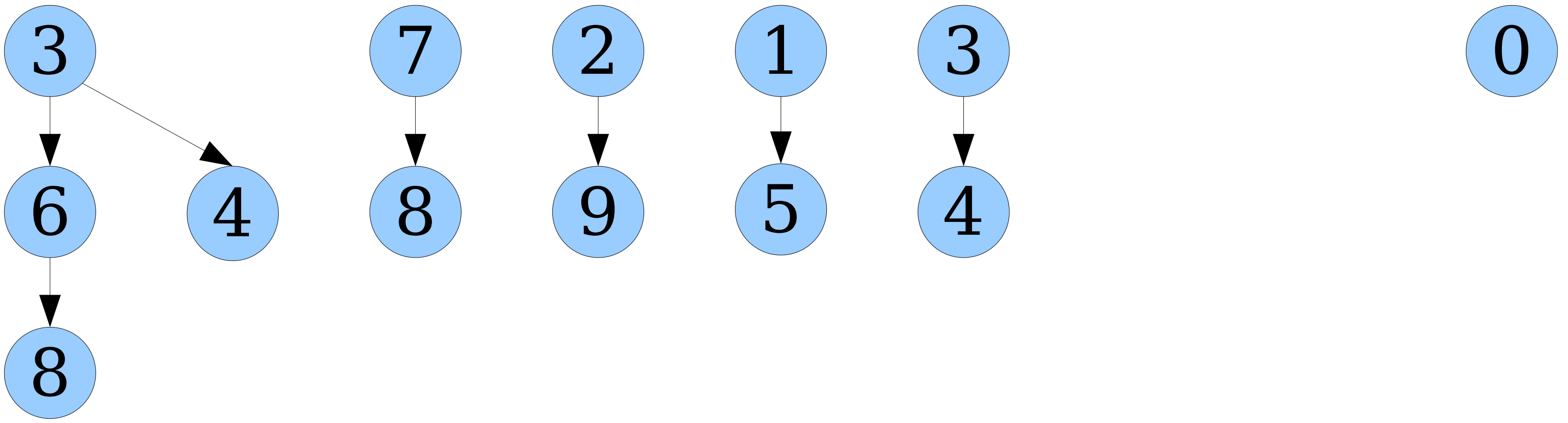
Coalescing Trees

- ***Observation:*** This would be a *lot* easier to do if all the trees were sorted by size.



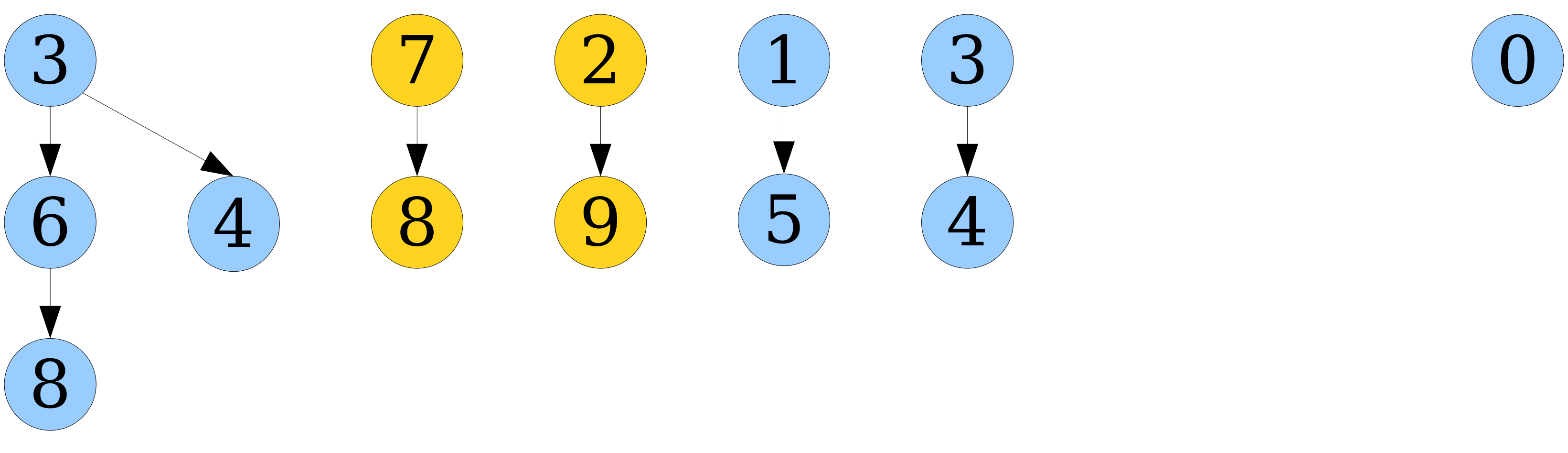
Coalescing Trees

- ***Observation:*** This would be a *lot* easier to do if all the trees were sorted by size.



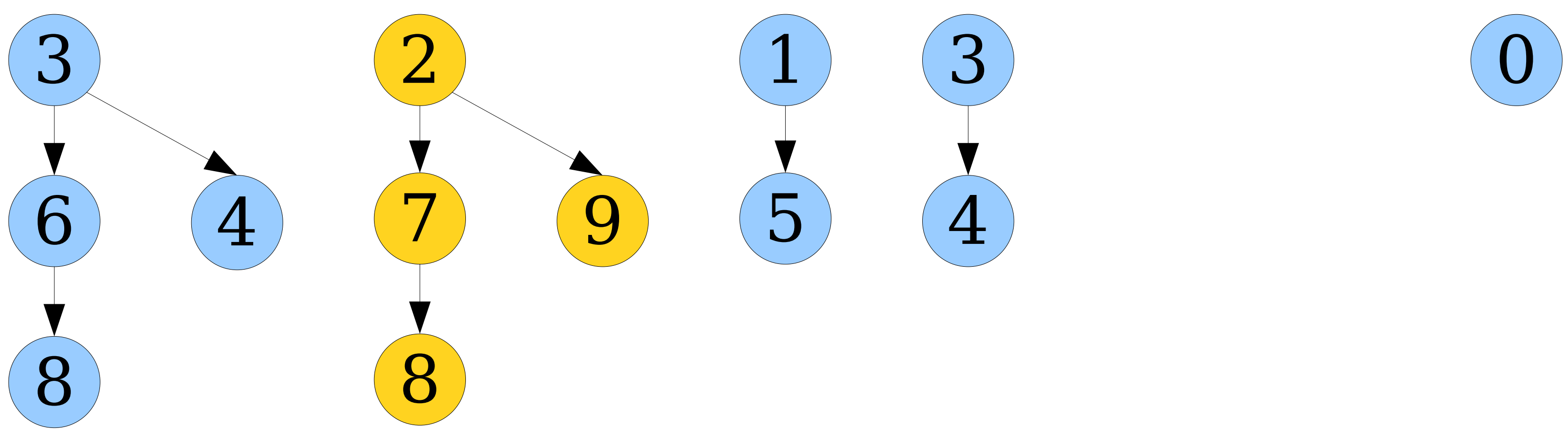
Coalescing Trees

- ***Observation:*** This would be a *lot* easier to do if all the trees were sorted by size.



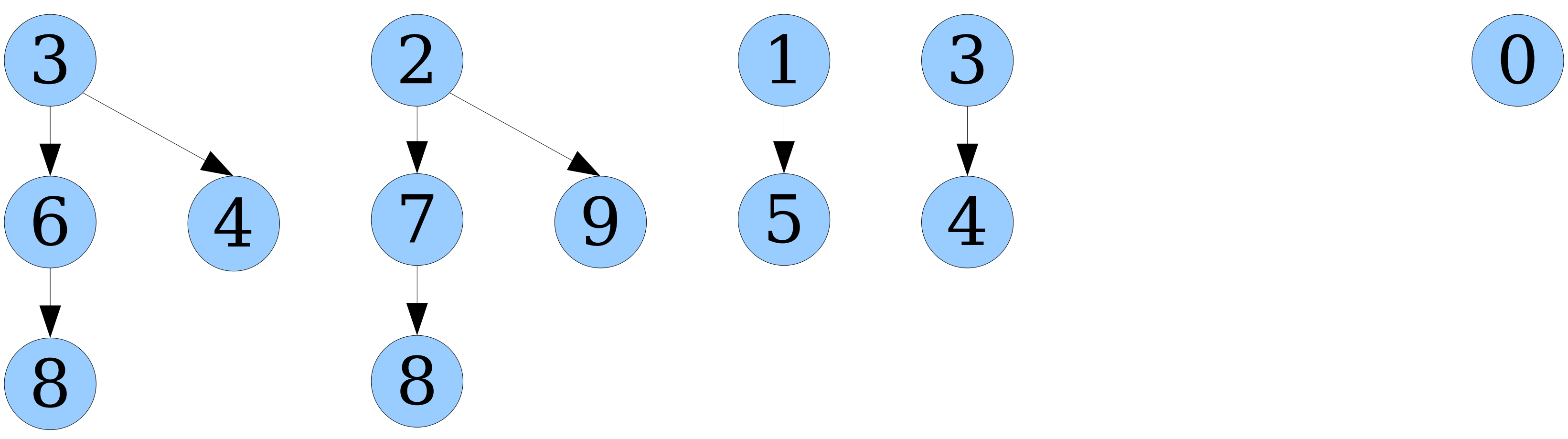
Coalescing Trees

- ***Observation:*** This would be a *lot* easier to do if all the trees were sorted by size.



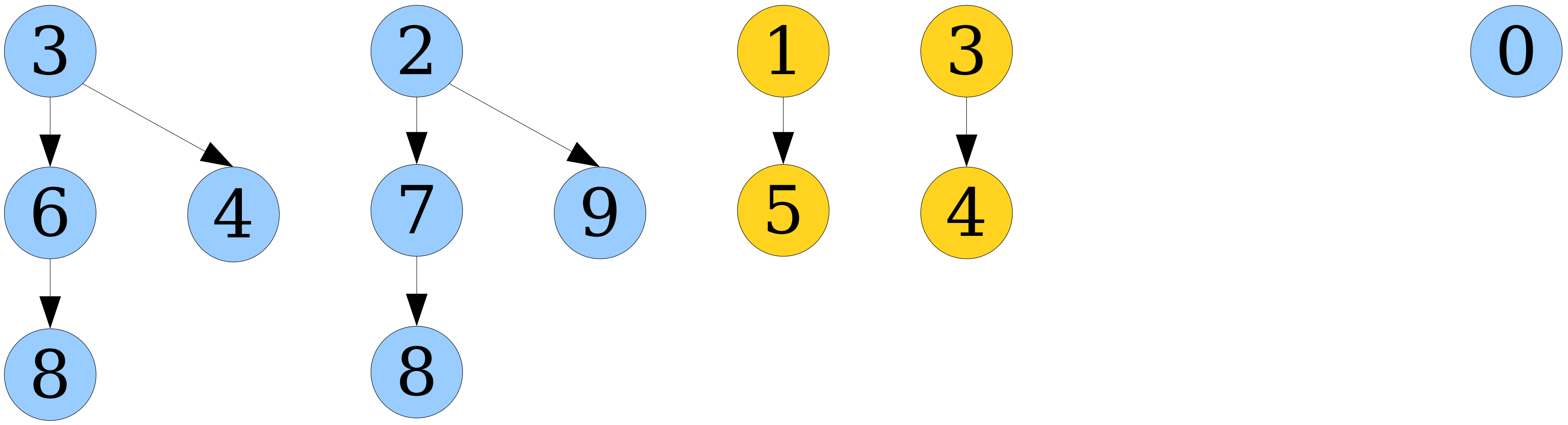
Coalescing Trees

- ***Observation:*** This would be a *lot* easier to do if all the trees were sorted by size.



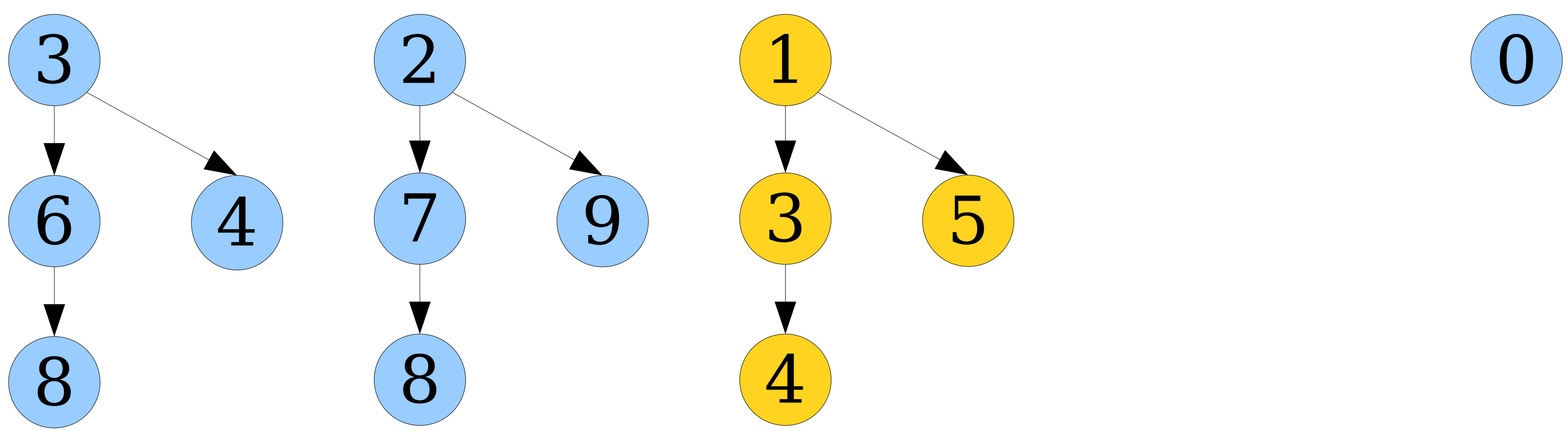
Coalescing Trees

- ***Observation:*** This would be a *lot* easier to do if all the trees were sorted by size.



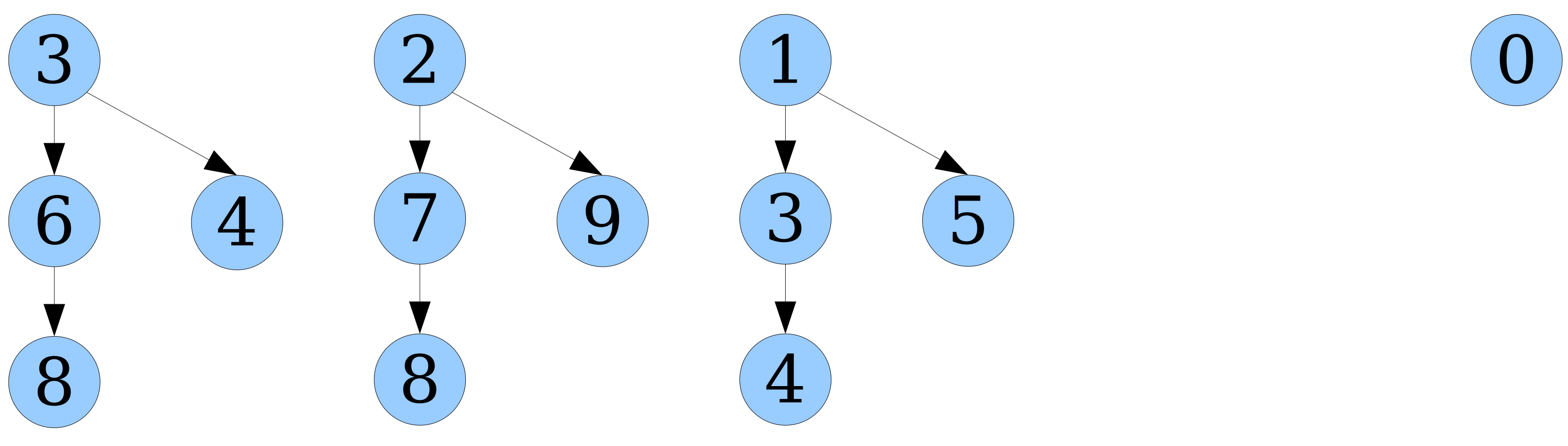
Coalescing Trees

- ***Observation:*** This would be a *lot* easier to do if all the trees were sorted by size.



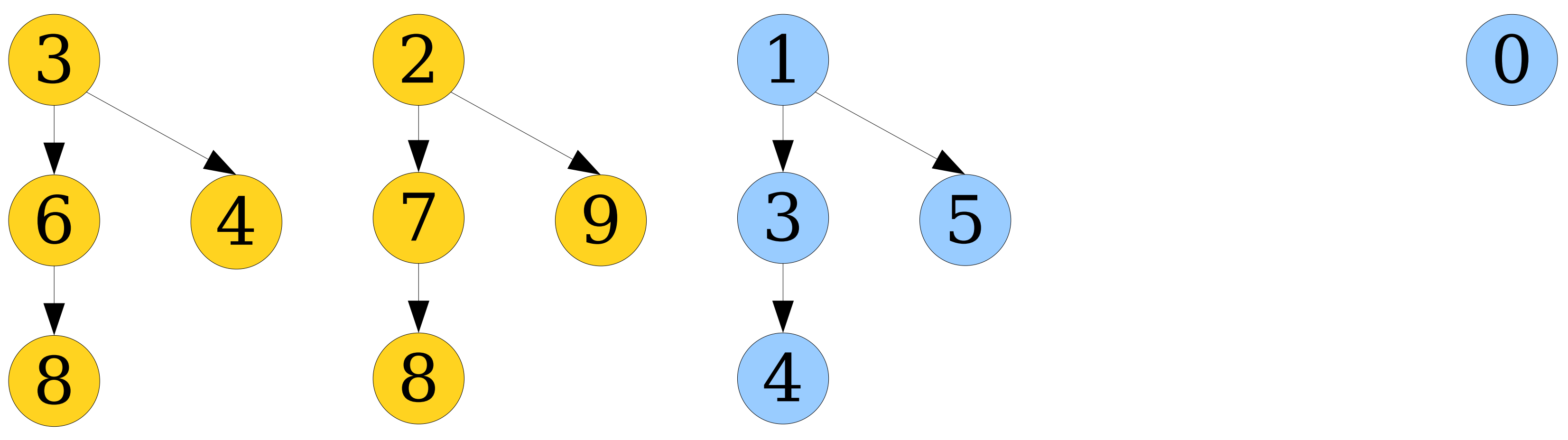
Coalescing Trees

- ***Observation:*** This would be a *lot* easier to do if all the trees were sorted by size.



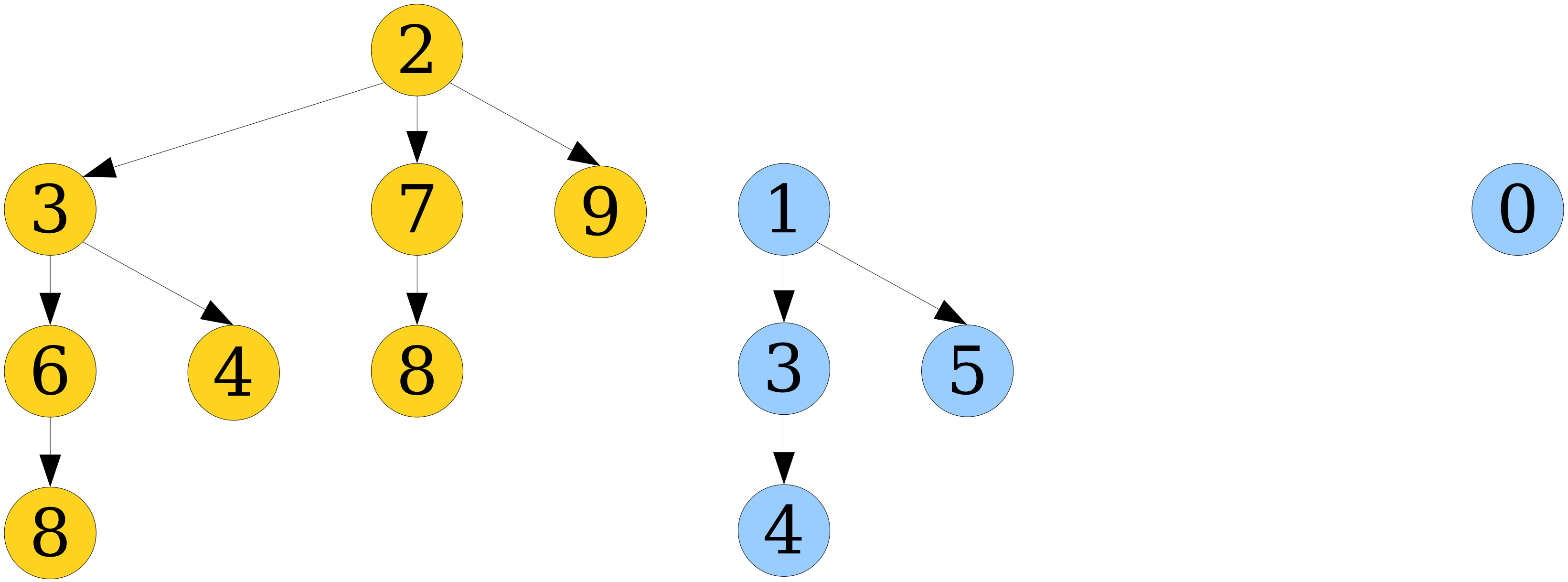
Coalescing Trees

- ***Observation:*** This would be a *lot* easier to do if all the trees were sorted by size.



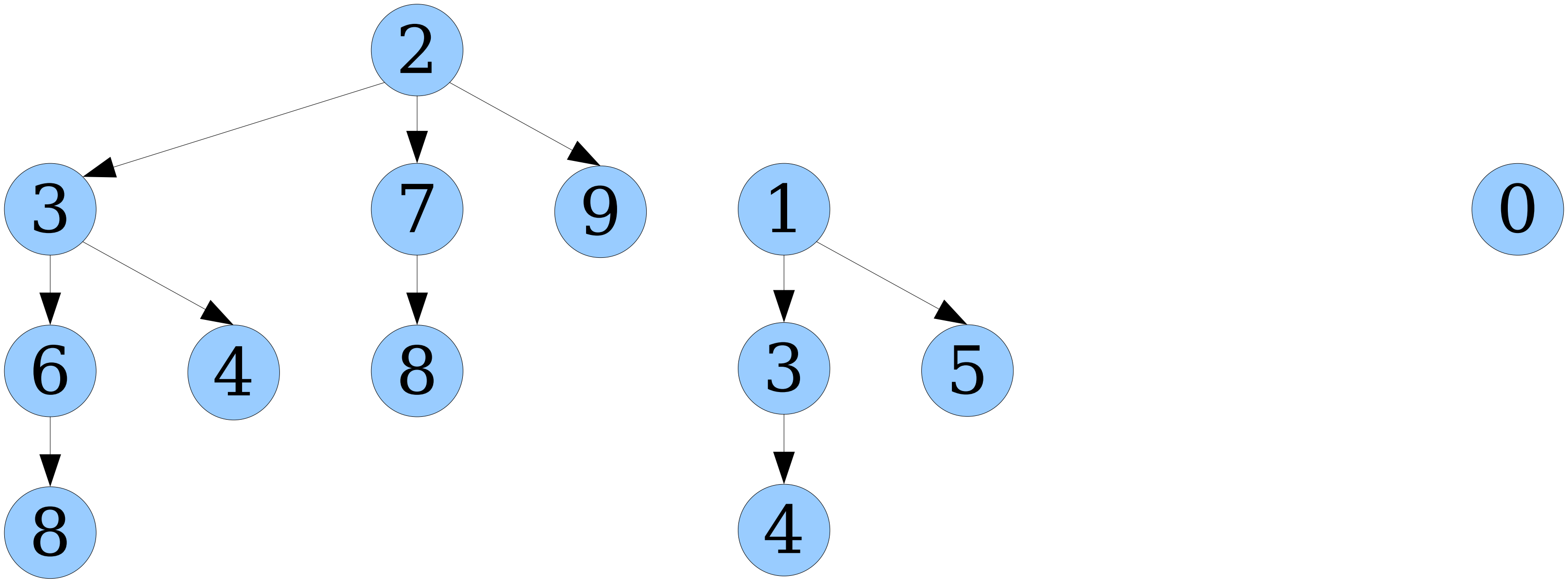
Coalescing Trees

- ***Observation:*** This would be a *lot* easier to do if all the trees were sorted by size.



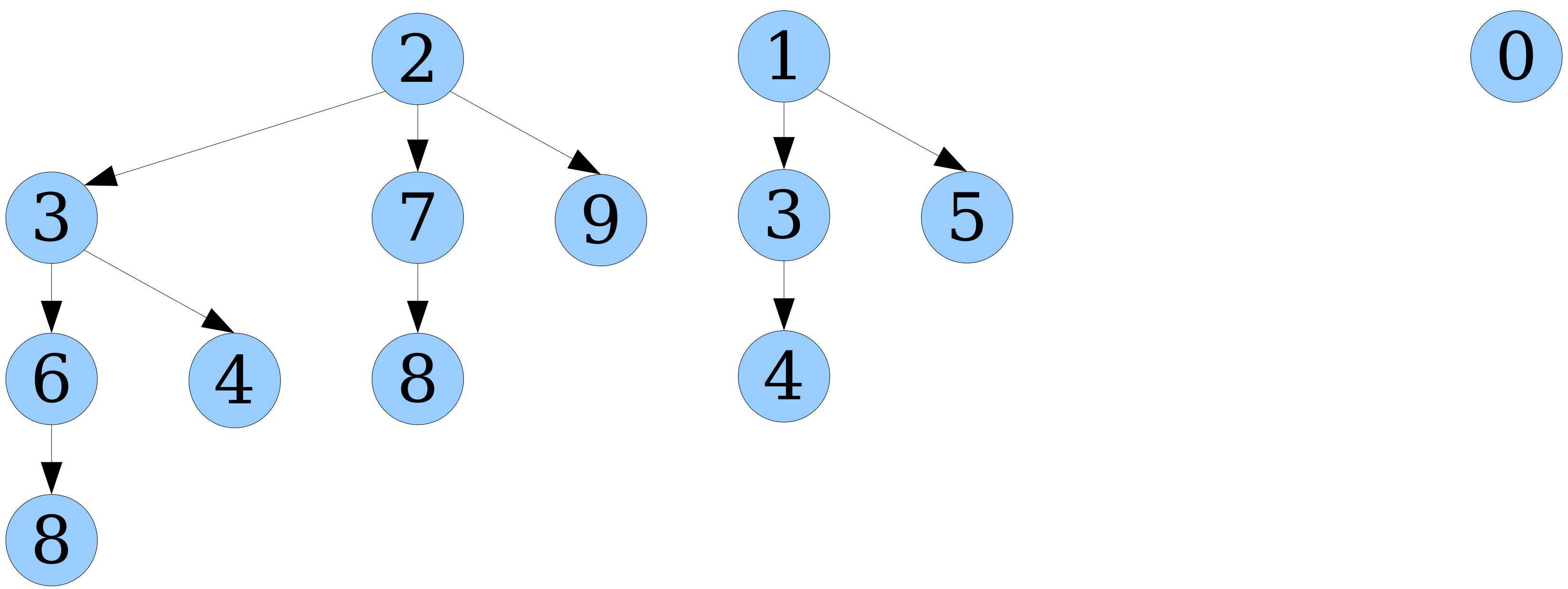
Coalescing Trees

- ***Observation:*** This would be a *lot* easier to do if all the trees were sorted by size.



Coalescing Trees

- ***Observation:*** This would be a *lot* easier to do if all the trees were sorted by size.

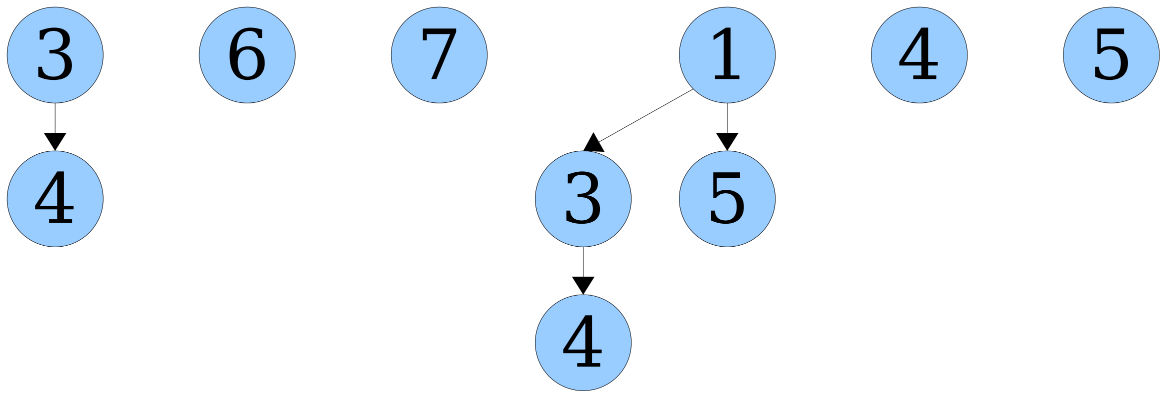


Coalescing Trees

- ***Observation:*** This would be a *lot* easier to do if all the trees were sorted by size.
- We can sort our group of t trees by size in time $O(t \log t)$ using a standard sorting algorithm.
- ***Better idea:*** All the sizes are small integers. Use counting sort!

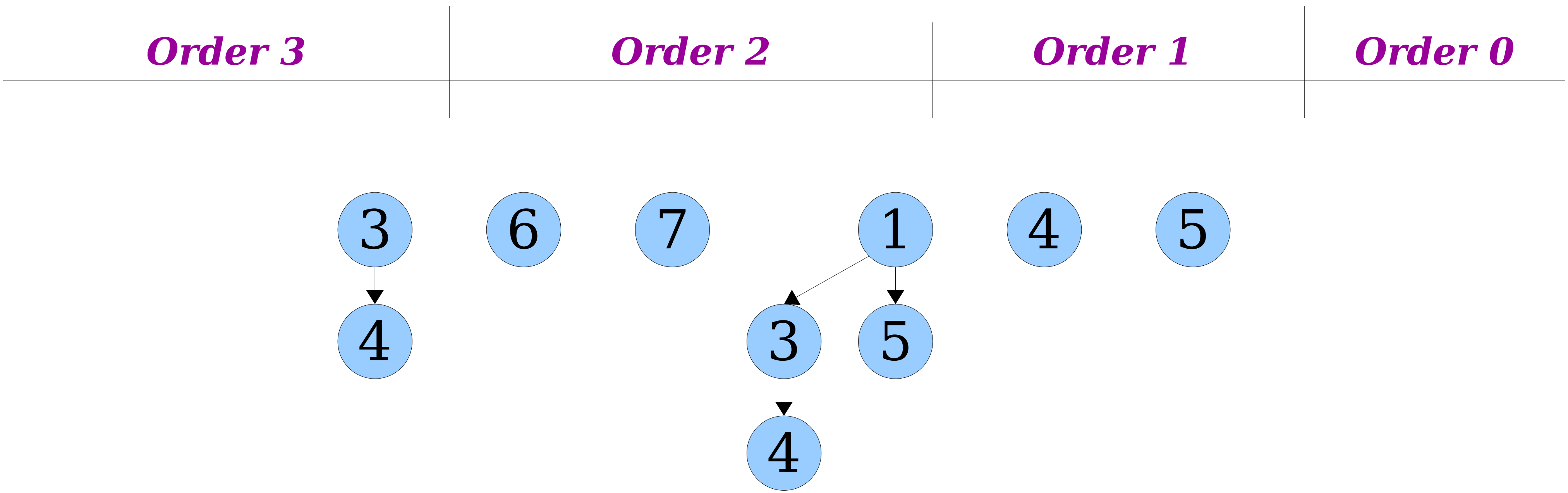
Coalescing Trees

- Here is a fast implementation of ***coalesce***:
  - Distribute the trees into an array of buckets big enough to hold trees of orders 0, 1, 2, …, $\lceil \log_2 (n + 1) \rceil$.
  - Start at bucket 0. While there's two or more trees in the bucket, fuse them and place the result one bucket higher.



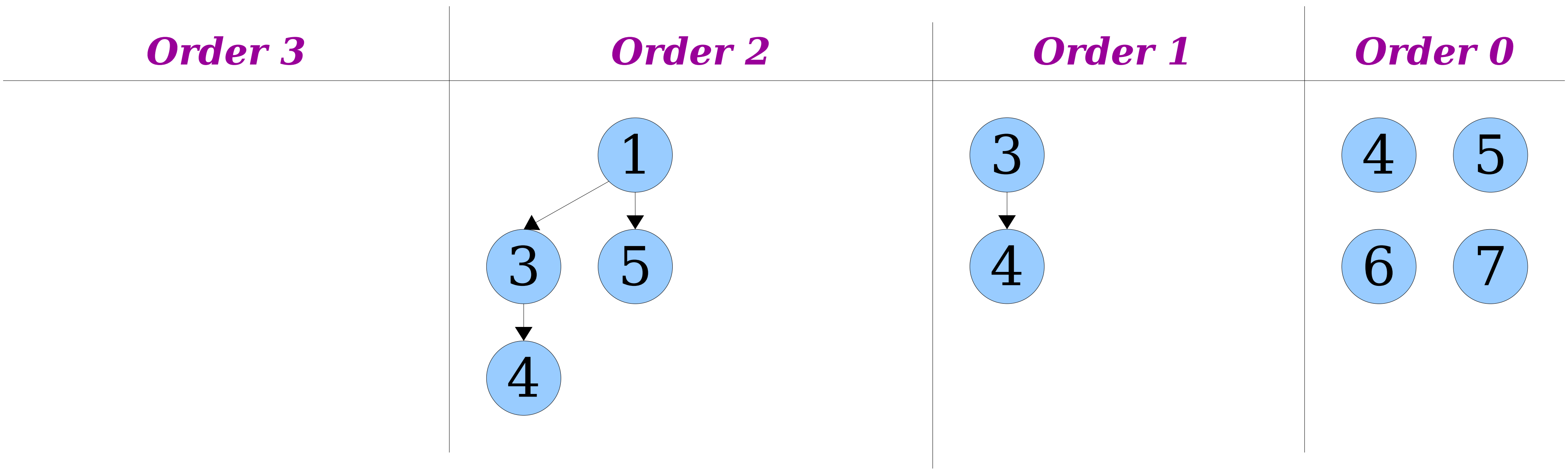
Coalescing Trees

- Here is a fast implementation of ***coalesce***:
  - Distribute the trees into an array of buckets big enough to hold trees of orders 0, 1, 2, …, $\lceil \log_2 (n + 1) \rceil$.
  - Start at bucket 0. While there's two or more trees in the bucket, fuse them and place the result one bucket higher.



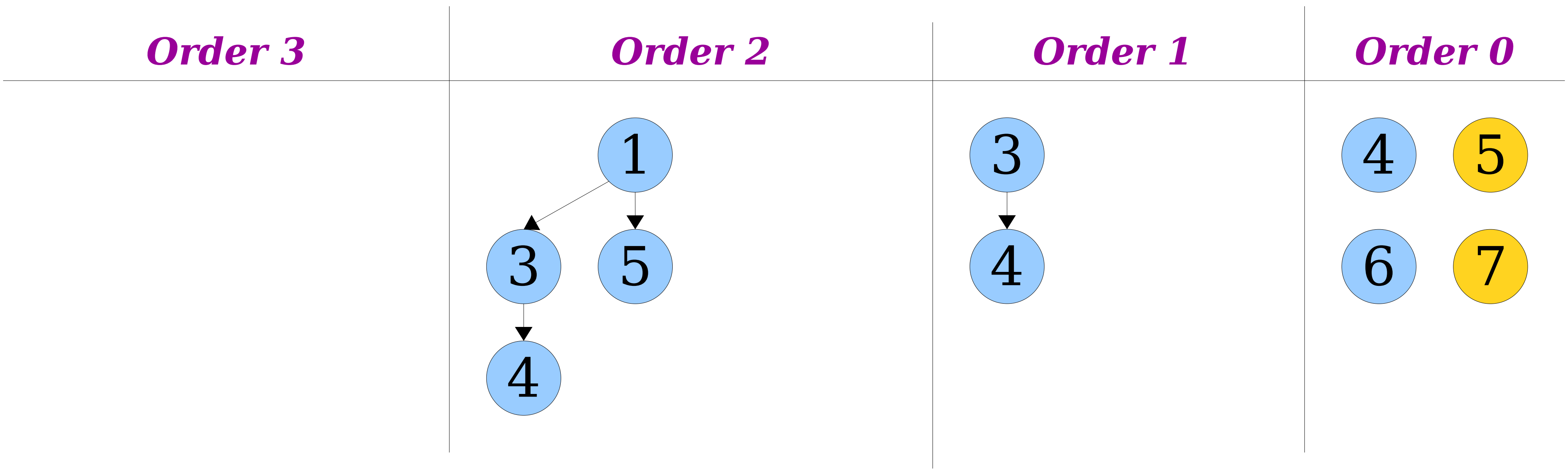
Coalescing Trees

- Here is a fast implementation of ***coalesce***:
  - Distribute the trees into an array of buckets big enough to hold trees of orders 0, 1, 2, …, $\lceil \log_2 (n + 1) \rceil$.
  - Start at bucket 0. While there’s two or more trees in the bucket, fuse them and place the result one bucket higher.



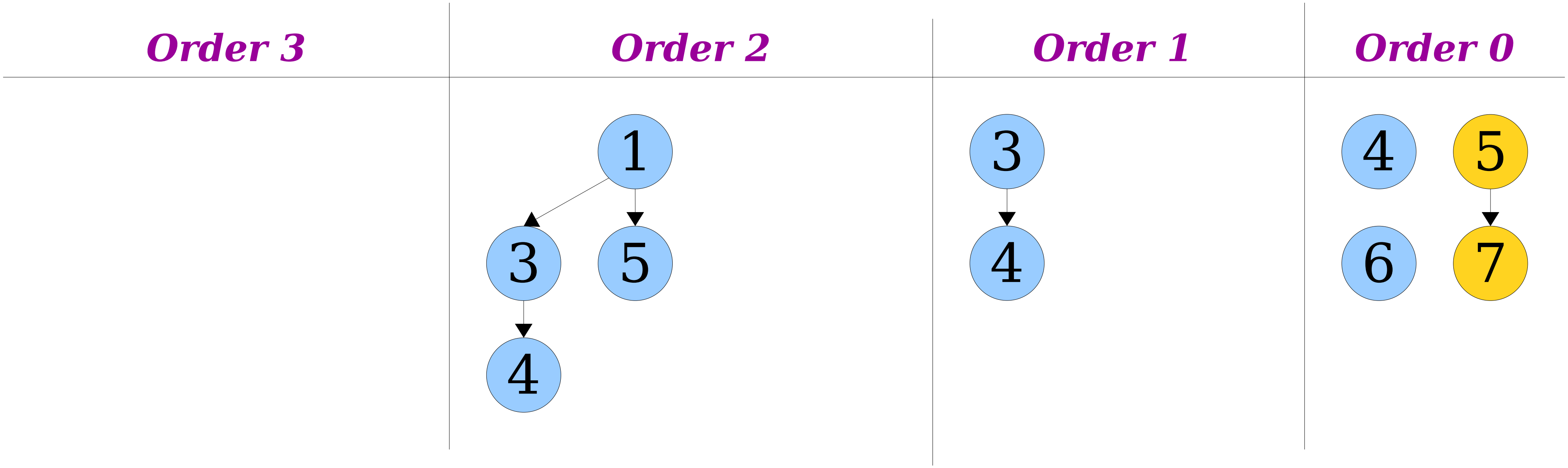
Coalescing Trees

- Here is a fast implementation of ***coalesce***:
  - Distribute the trees into an array of buckets big enough to hold trees of orders 0, 1, 2, …, $\lceil \log_2 (n + 1) \rceil$.
  - Start at bucket 0. While there’s two or more trees in the bucket, fuse them and place the result one bucket higher.



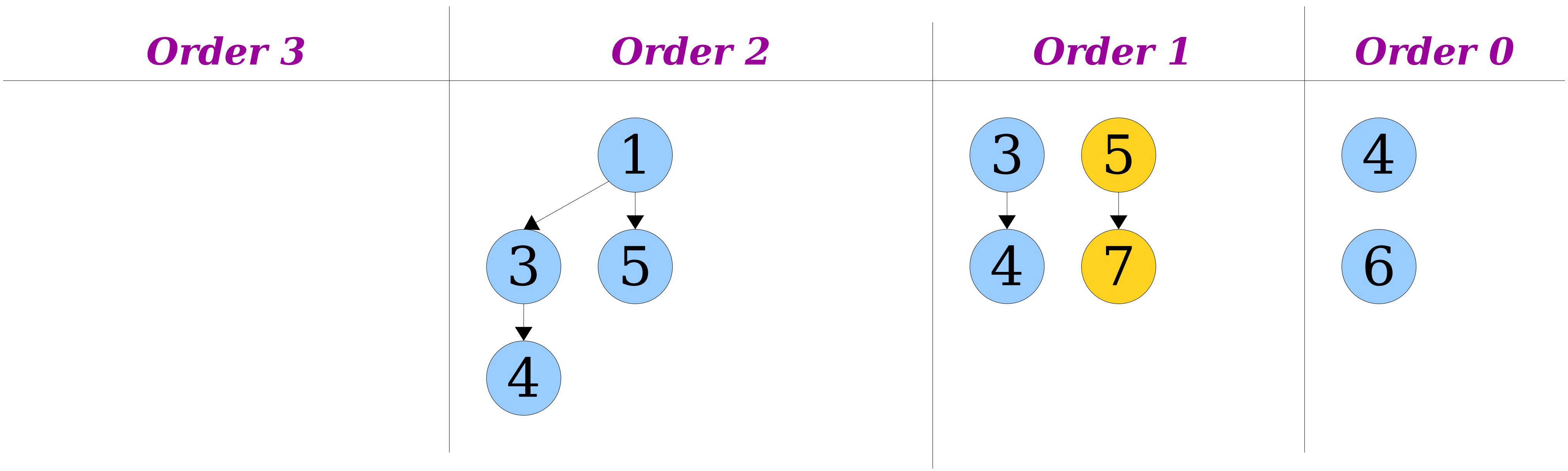
Coalescing Trees

- Here is a fast implementation of ***coalesce***:
  - Distribute the trees into an array of buckets big enough to hold trees of orders 0, 1, 2, …, $\lceil \log_2 (n + 1) \rceil$.
  - Start at bucket 0. While there's two or more trees in the bucket, fuse them and place the result one bucket higher.



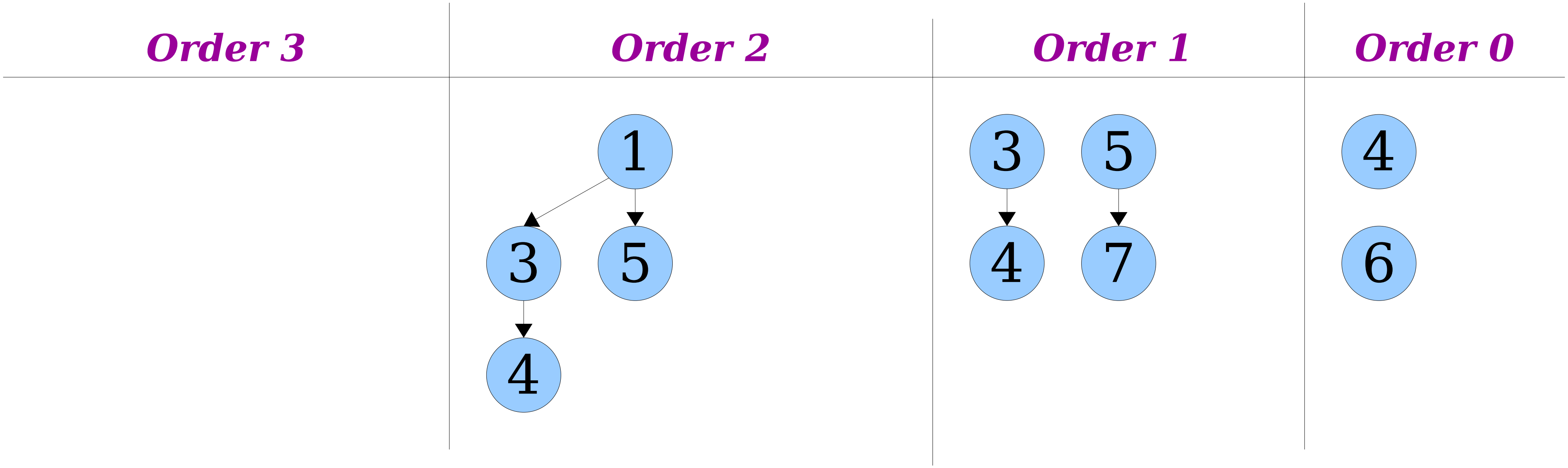
Coalescing Trees

- Here is a fast implementation of ***coalesce***:
  - Distribute the trees into an array of buckets big enough to hold trees of orders 0, 1, 2, …, $\lceil \log_2 (n + 1) \rceil$.
  - Start at bucket 0. While there's two or more trees in the bucket, fuse them and place the result one bucket higher.



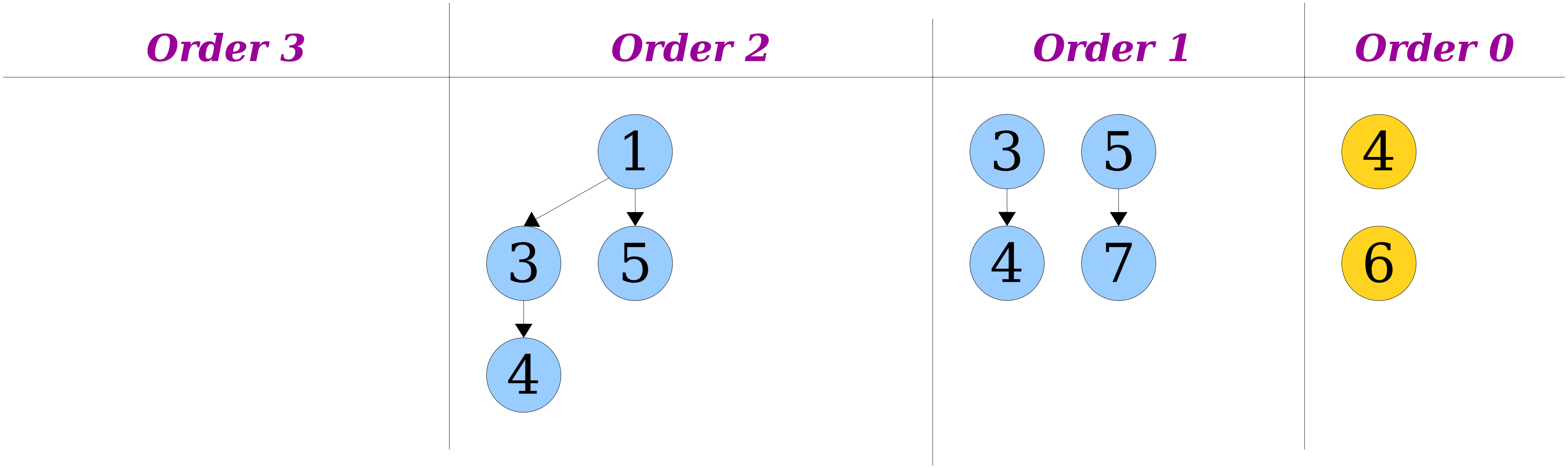
Coalescing Trees

- Here is a fast implementation of ***coalesce***:
  - Distribute the trees into an array of buckets big enough to hold trees of orders 0, 1, 2, …, $\lceil \log_2 (n + 1) \rceil$.
  - Start at bucket 0. While there's two or more trees in the bucket, fuse them and place the result one bucket higher.



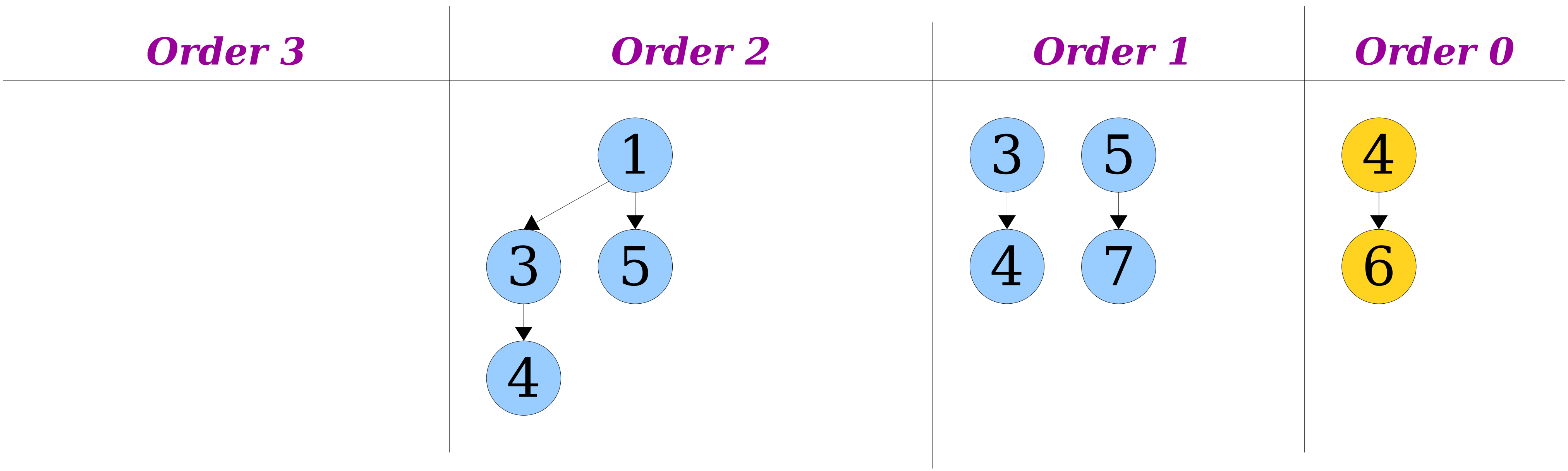
Coalescing Trees

- Here is a fast implementation of ***coalesce***:
  - Distribute the trees into an array of buckets big enough to hold trees of orders 0, 1, 2, …, $\lceil \log_2 (n + 1) \rceil$.
  - Start at bucket 0. While there's two or more trees in the bucket, fuse them and place the result one bucket higher.



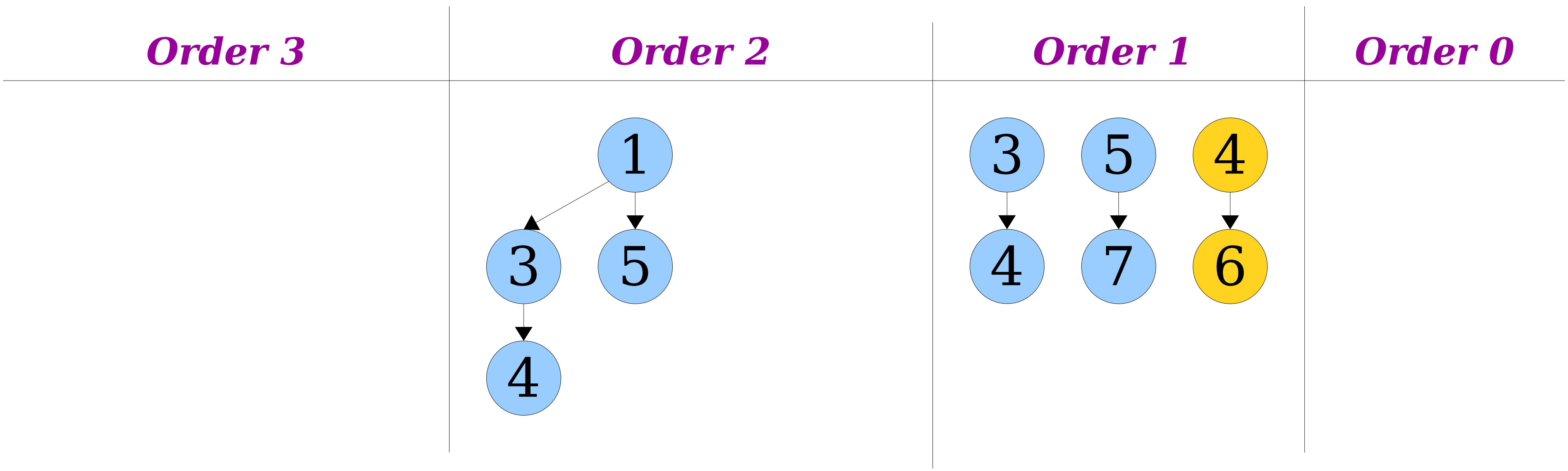
Coalescing Trees

- Here is a fast implementation of ***coalesce***:
  - Distribute the trees into an array of buckets big enough to hold trees of orders 0, 1, 2, …, $\lceil \log_2 (n + 1) \rceil$.
  - Start at bucket 0. While there's two or more trees in the bucket, fuse them and place the result one bucket higher.



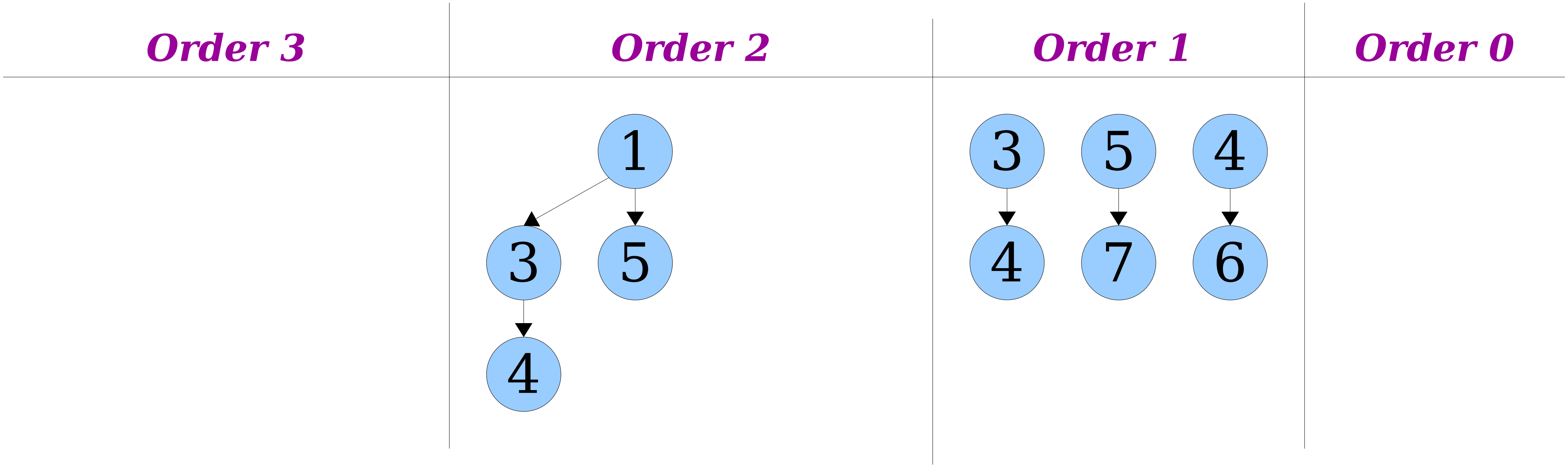
Coalescing Trees

- Here is a fast implementation of ***coalesce***:
  - Distribute the trees into an array of buckets big enough to hold trees of orders 0, 1, 2, …, $\lceil \log_2 (n + 1) \rceil$.
  - Start at bucket 0. While there's two or more trees in the bucket, fuse them and place the result one bucket higher.



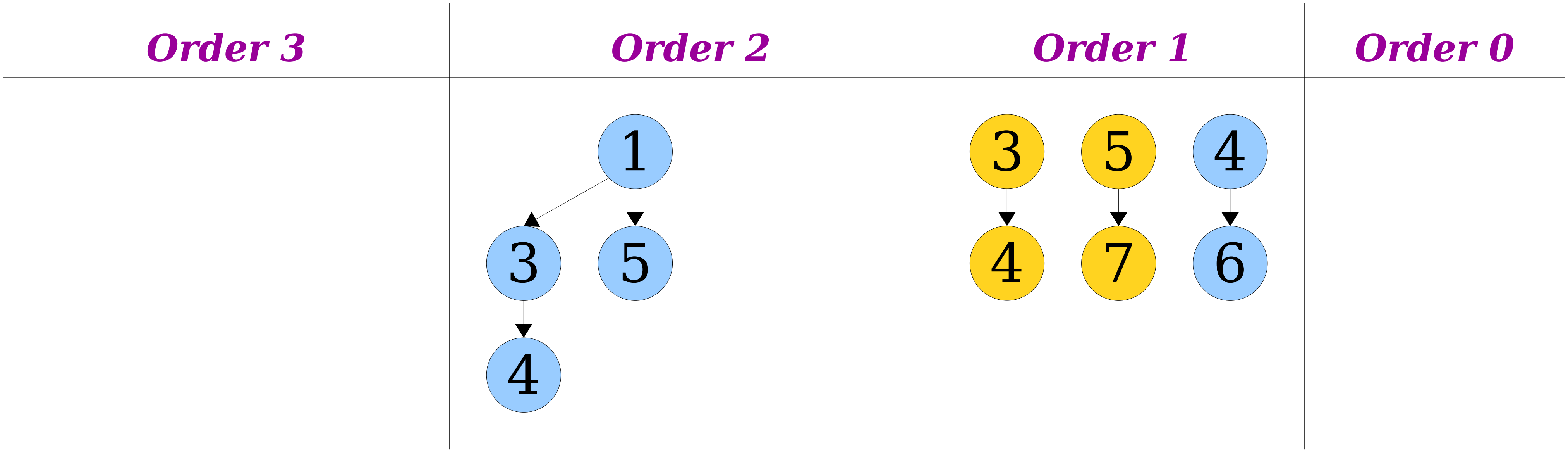
Coalescing Trees

- Here is a fast implementation of ***coalesce***:
  - Distribute the trees into an array of buckets big enough to hold trees of orders 0, 1, 2, …, $\lceil \log_2 (n + 1) \rceil$.
  - Start at bucket 0. While there's two or more trees in the bucket, fuse them and place the result one bucket higher.



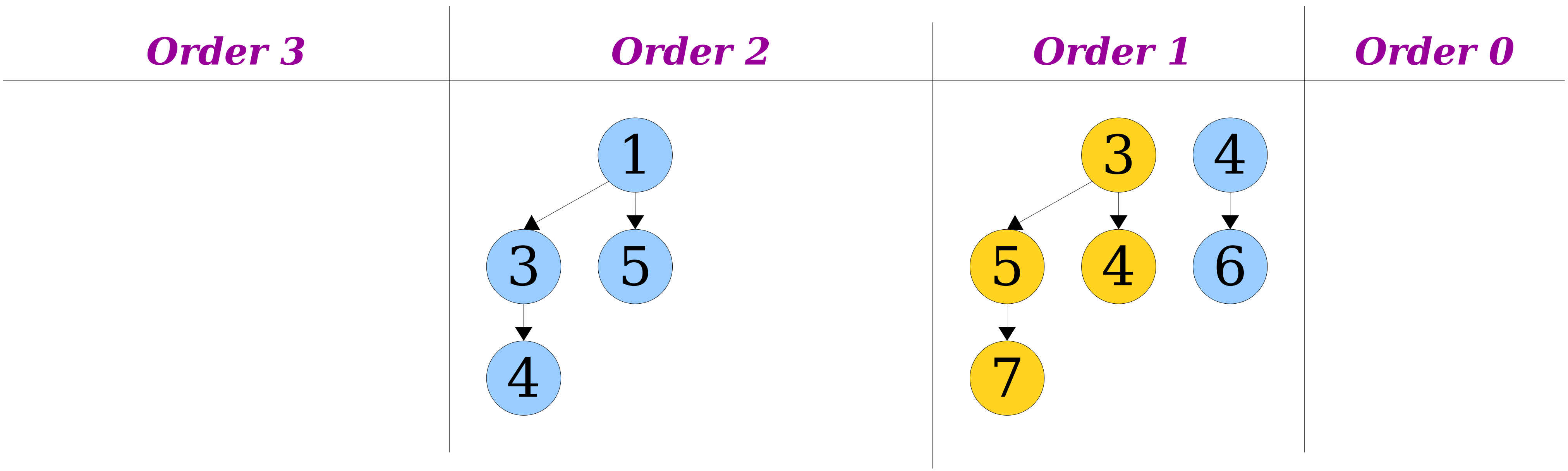
Coalescing Trees

- Here is a fast implementation of ***coalesce***:
  - Distribute the trees into an array of buckets big enough to hold trees of orders 0, 1, 2, …, $\lceil \log_2 (n + 1) \rceil$.
  - Start at bucket 0. While there's two or more trees in the bucket, fuse them and place the result one bucket higher.



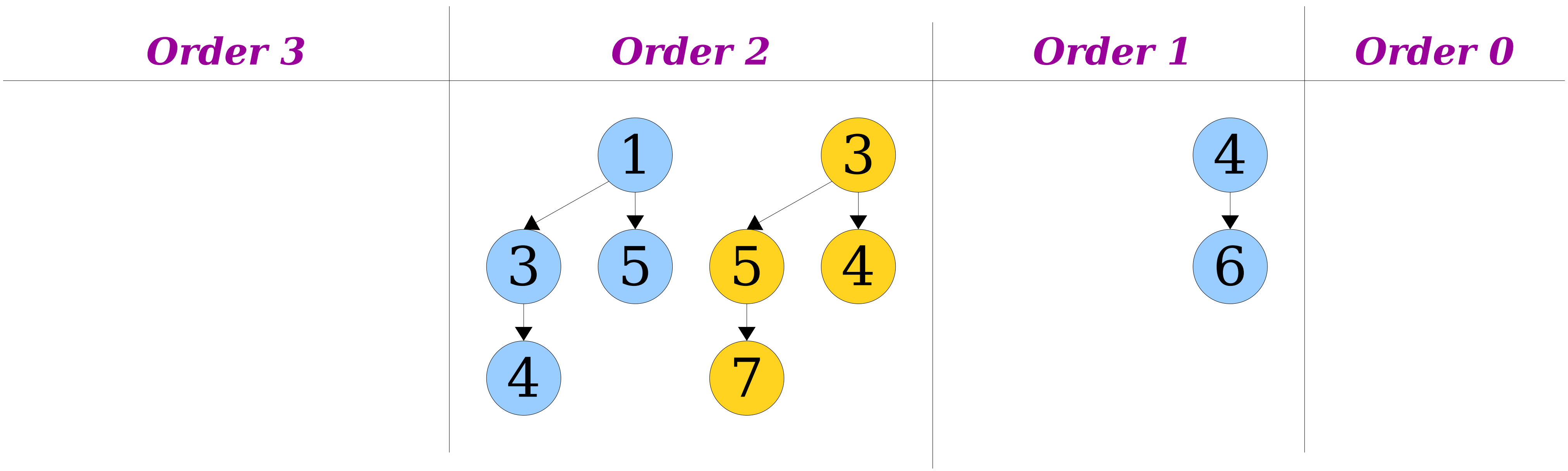
Coalescing Trees

- Here is a fast implementation of ***coalesce***:
  - Distribute the trees into an array of buckets big enough to hold trees of orders 0, 1, 2, …, $\lceil \log_2 (n + 1) \rceil$.
  - Start at bucket 0. While there's two or more trees in the bucket, fuse them and place the result one bucket higher.



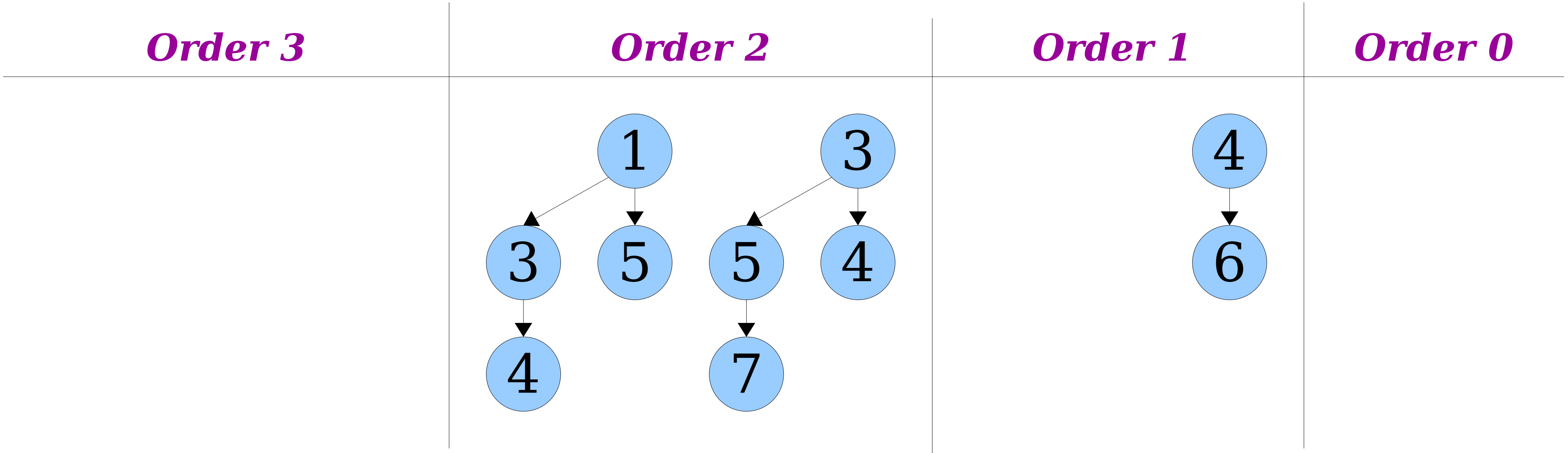
Coalescing Trees

- Here is a fast implementation of ***coalesce***:
  - Distribute the trees into an array of buckets big enough to hold trees of orders 0, 1, 2, …, $\lceil \log_2 (n + 1) \rceil$.
  - Start at bucket 0. While there's two or more trees in the bucket, fuse them and place the result one bucket higher.



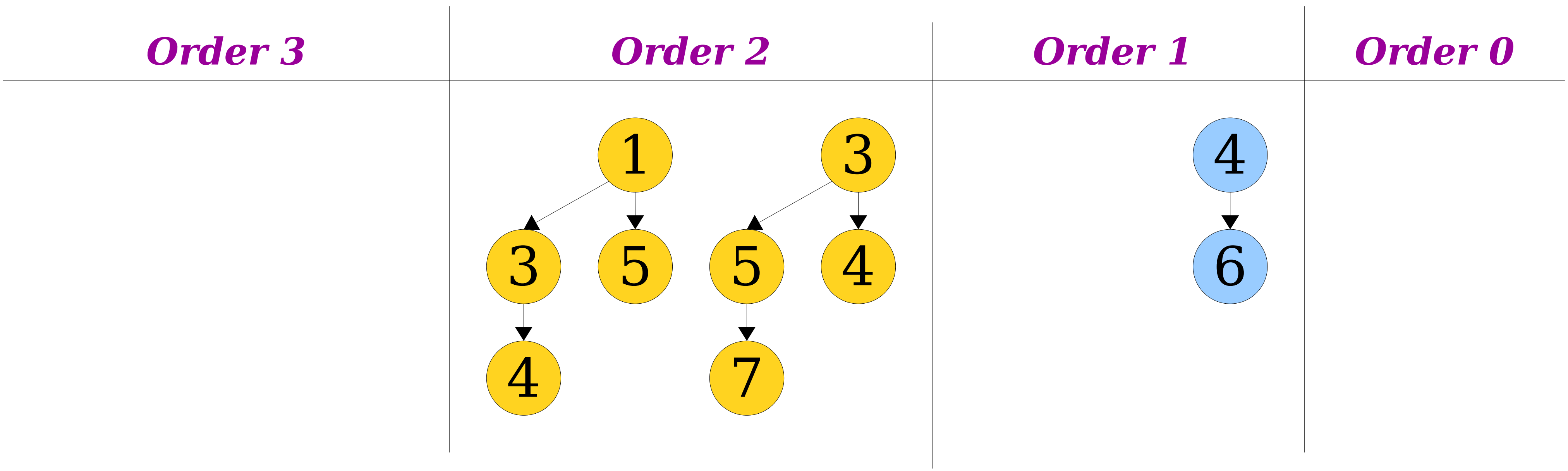
Coalescing Trees

- Here is a fast implementation of ***coalesce***:
  - Distribute the trees into an array of buckets big enough to hold trees of orders 0, 1, 2, …, $\lceil \log_2 (n + 1) \rceil$.
  - Start at bucket 0. While there's two or more trees in the bucket, fuse them and place the result one bucket higher.



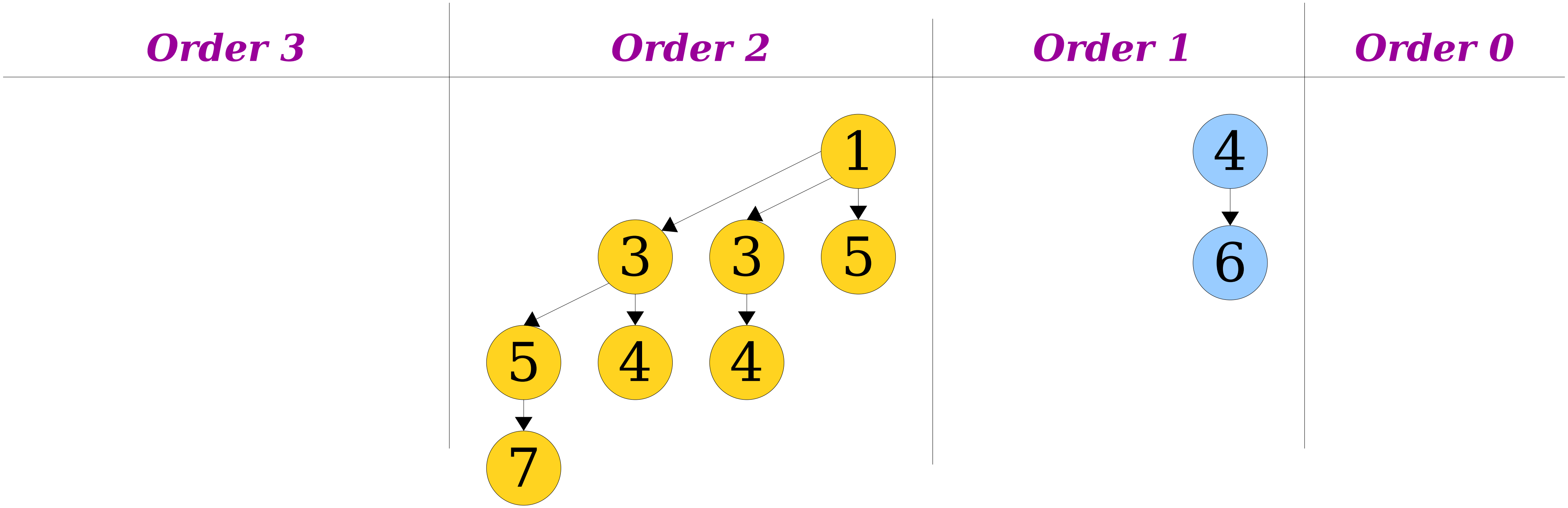
Coalescing Trees

- Here is a fast implementation of ***coalesce***:
  - Distribute the trees into an array of buckets big enough to hold trees of orders 0, 1, 2, …, $\lceil \log_2 (n + 1) \rceil$.
  - Start at bucket 0. While there's two or more trees in the bucket, fuse them and place the result one bucket higher.



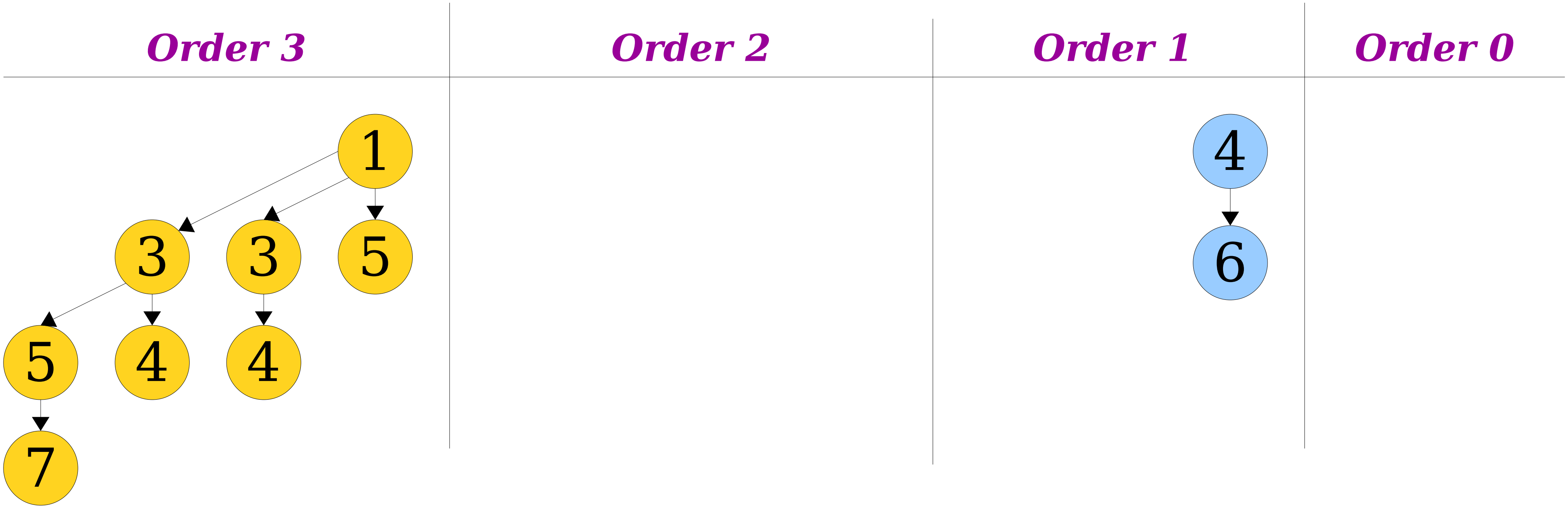
Coalescing Trees

- Here is a fast implementation of ***coalesce***:
  - Distribute the trees into an array of buckets big enough to hold trees of orders 0, 1, 2, …, $\lceil \log_2 (n + 1) \rceil$.
  - Start at bucket 0. While there's two or more trees in the bucket, fuse them and place the result one bucket higher.



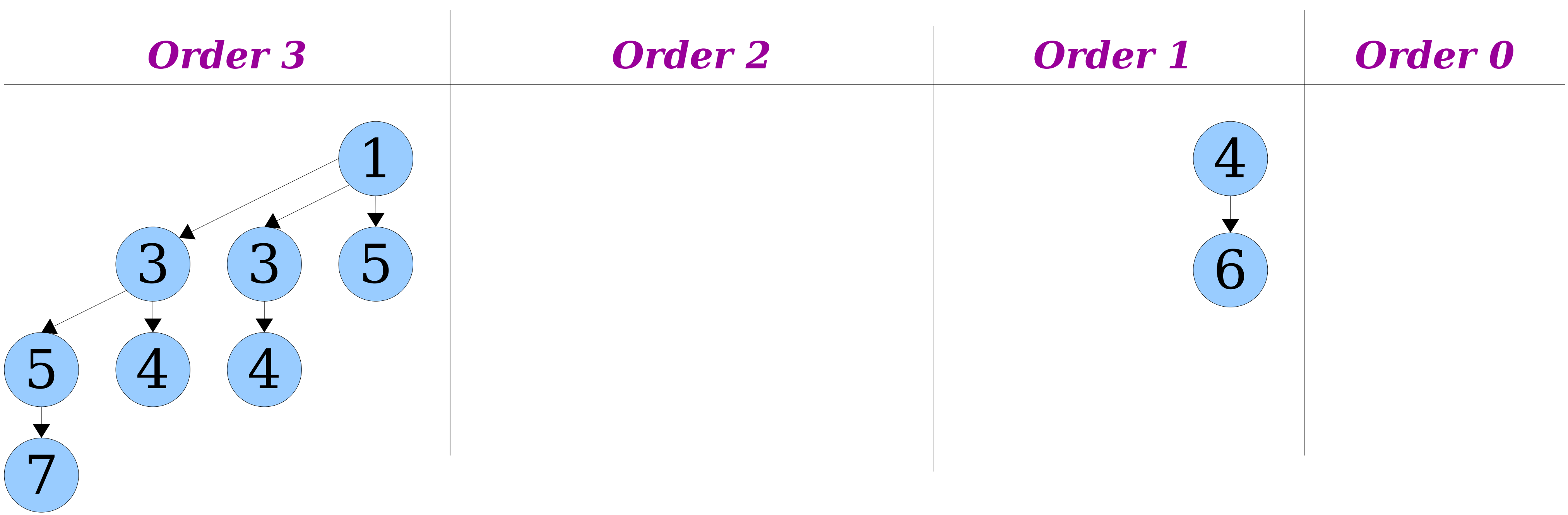
Coalescing Trees

- Here is a fast implementation of ***coalesce***:
  - Distribute the trees into an array of buckets big enough to hold trees of orders 0, 1, 2, …, $\lceil \log_2 (n + 1) \rceil$.
  - Start at bucket 0. While there's two or more trees in the bucket, fuse them and place the result one bucket higher.



Coalescing Trees

- Here is a fast implementation of ***coalesce***:
  - Distribute the trees into an array of buckets big enough to hold trees of orders 0, 1, 2, …, $\lceil \log_2 (n + 1) \rceil$.
  - Start at bucket 0. While there's two or more trees in the bucket, fuse them and place the result one bucket higher.



Coalescing Trees

- Here is a fast implementation of ***coalesce***:
  - Distribute the trees into an array of buckets big enough to hold trees of orders 0, 1, 2, …, $\lceil \log_2 (n + 1) \rceil$.
  - Start at bucket 0. While there's two or more trees in the bucket, fuse them and place the result one bucket higher.

Analyzing Coalesce

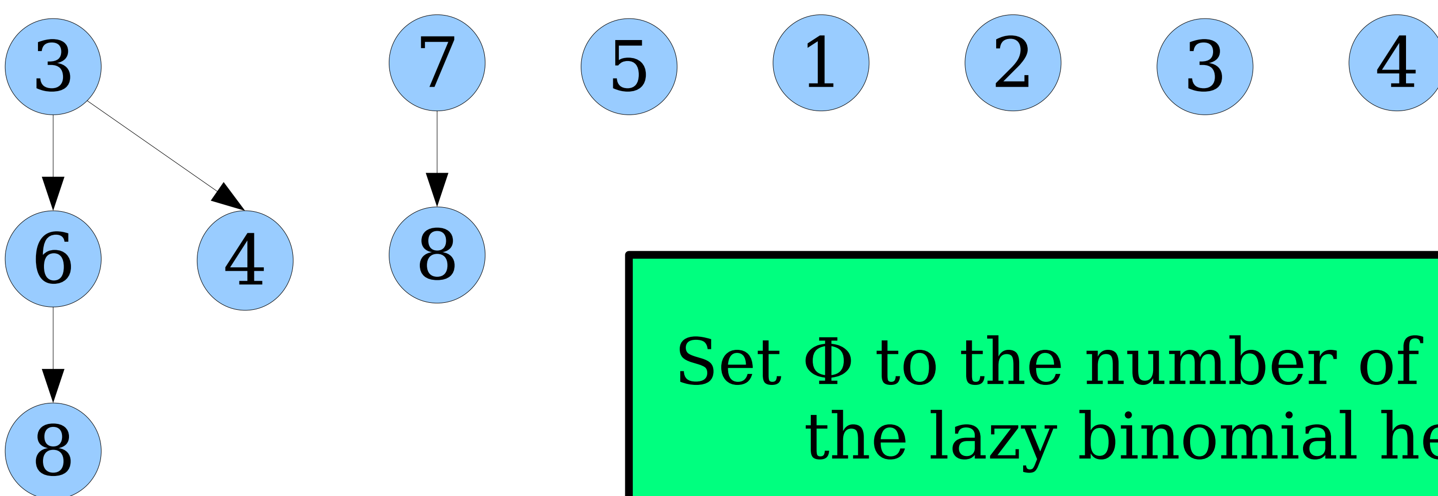
- How much time does it take to coalesce a group of t trees?
  - Time to create the array of buckets: $O(\log n)$.
  - Time to distribute trees into buckets: $O(t)$.
  - Time to fuse trees: $O(t + \log n)$
    - Number of fuses is $O(t)$, since each fuse decreases the number of trees by one. Cost per fuse is $O(1)$.
    - Need to iterate across $O(\log n)$ buckets.
- Total work done: $\mathbf{O(t + \log n)}$.
- In the worst case, this is $O(n)$.

The Story So Far

- A binomial heap with lazy melding has these worst-case time bounds:
  - ***enqueue***: O(1)
  - ***meld***: O(1)
  - ***find-min***: O(1)
  - ***extract-min***: O(n).
- But these are *worst-case* time bounds. Intuitively, things should nicely amortize away.
  - The number of trees grows slowly (one per ***enqueue***).
  - The number of trees drops quickly (at most one tree per order) after an ***extract-min***).

An Amortized Analysis

- This is a great spot to use an amortized analysis by defining a potential function Φ.
- In each case, the idea is to clearly mark what “messes” we need to clean up.
- In our case, each tree is a “mess,” since our future ***coalesce*** operation has to clean it up.



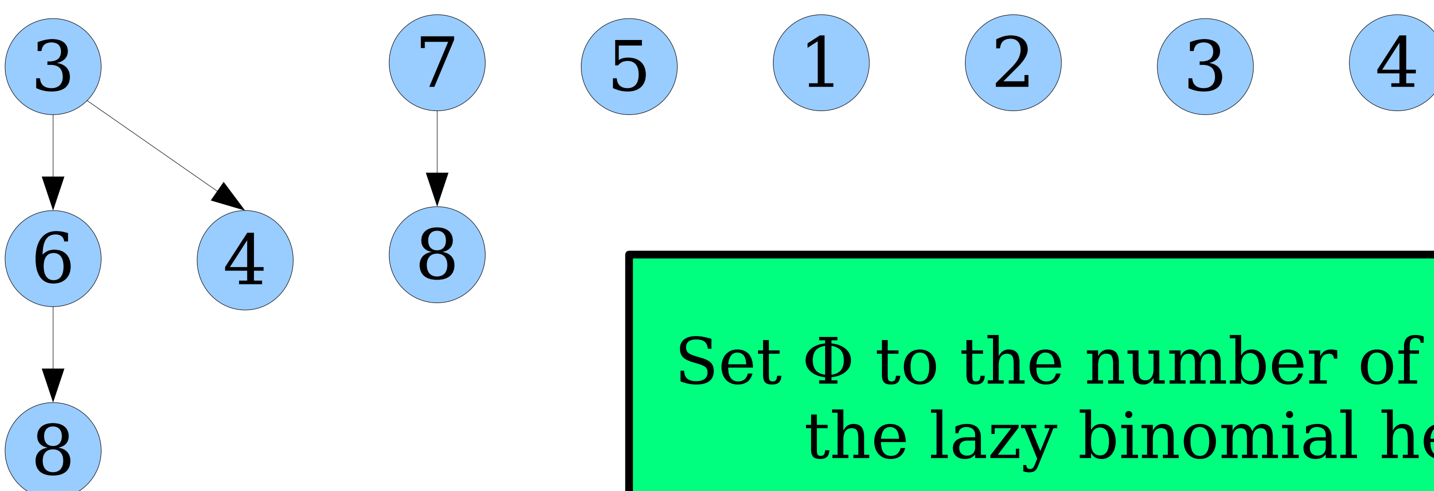
Set Φ to the number of trees in the lazy binomial heap.

An Amortized Analysis

- ***Recall:*** We assign amortized costs as

  $$\textbf{\textit{amortized-cost}} = \textbf{\textit{real-cost}} + \boldsymbol{k} \cdot \boldsymbol{\Delta\Phi},$$

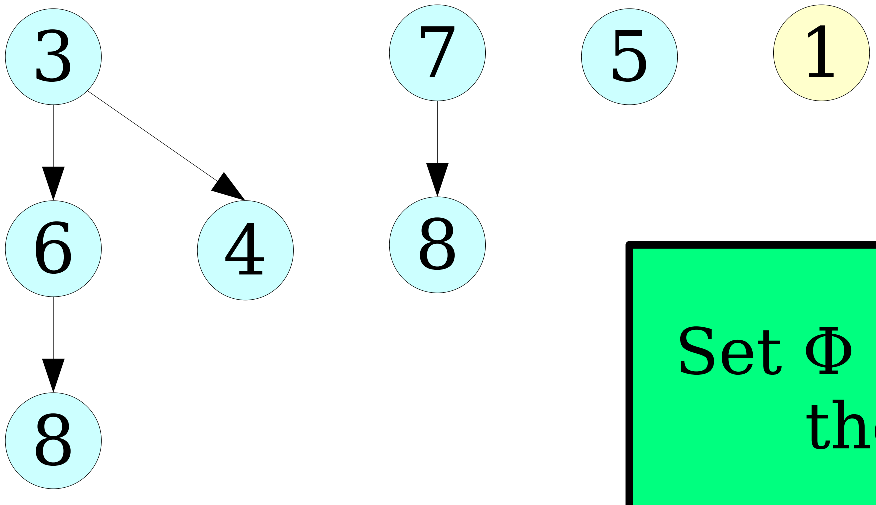
  where $\Delta\Phi = \Phi_{after} - \Phi_{before}$.
  - Increasing Φ (adding more trees) artificially boosts costs.
  - Decreasing Φ (removing trees) artificially lowers costs.
- Let's work out the amortized costs of each operation on a lazy binomial heap.



Set Φ to the number of trees in the lazy binomial heap.

Analyzing an Insertion

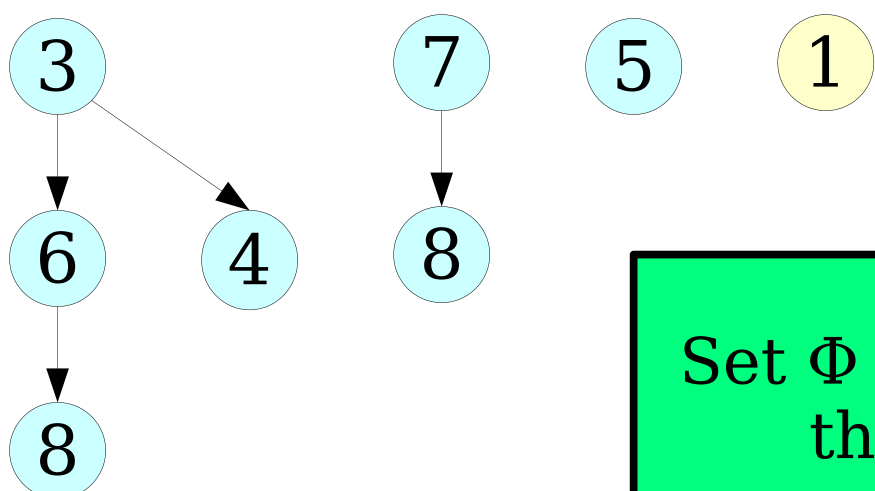
- To ***enqueue*** a key, we add a new binomial tree to the forest.



Set Φ to the number of trees in the lazy binomial heap.

Analyzing an Insertion

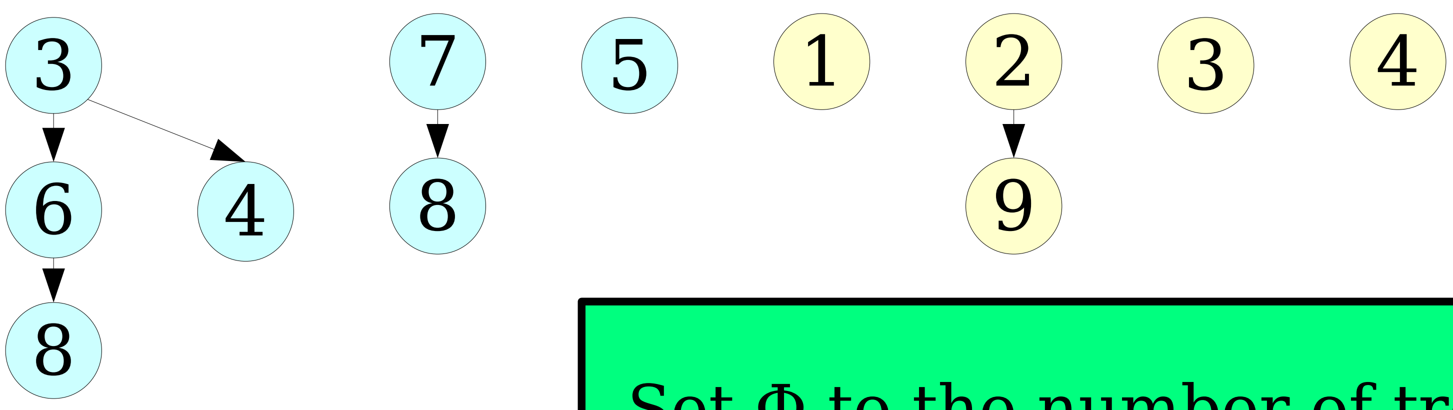
- To ***enqueue*** a key, we add a new binomial tree to the forest.
- Real cost: O(1). ΔΦ: +1
- Amortized cost: **O(1)**.



Set Φ to the number of trees in the lazy binomial heap.

Analyzing a Meld

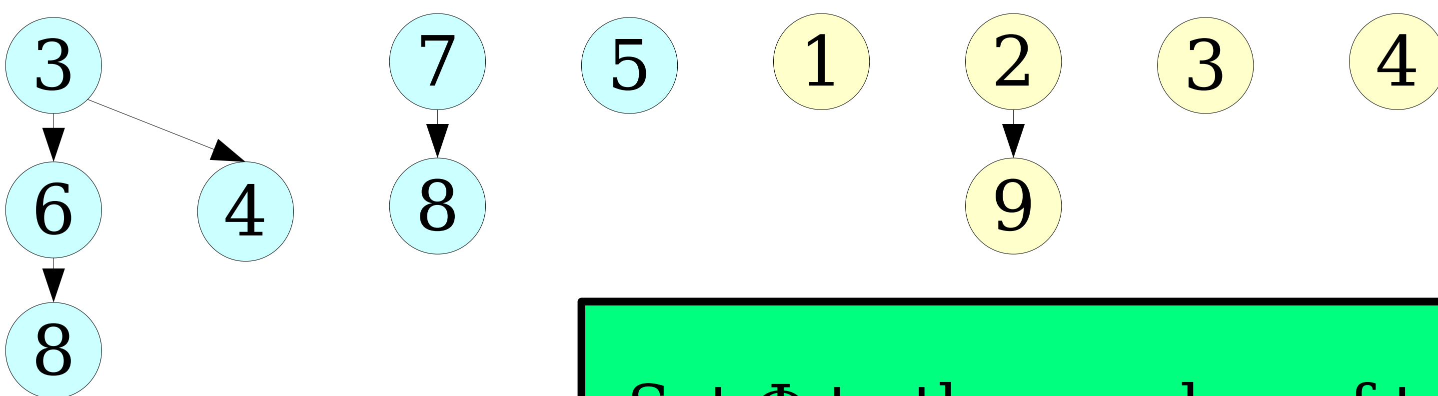
- What is the amortized cost of ***meld***?
- The real cost is O(1).
- What's $\Delta\Phi$?
- That's trickier – there are two separate collections of trees here.



Set Φ to the number of trees in the lazy binomial heap.

Analyzing a Meld

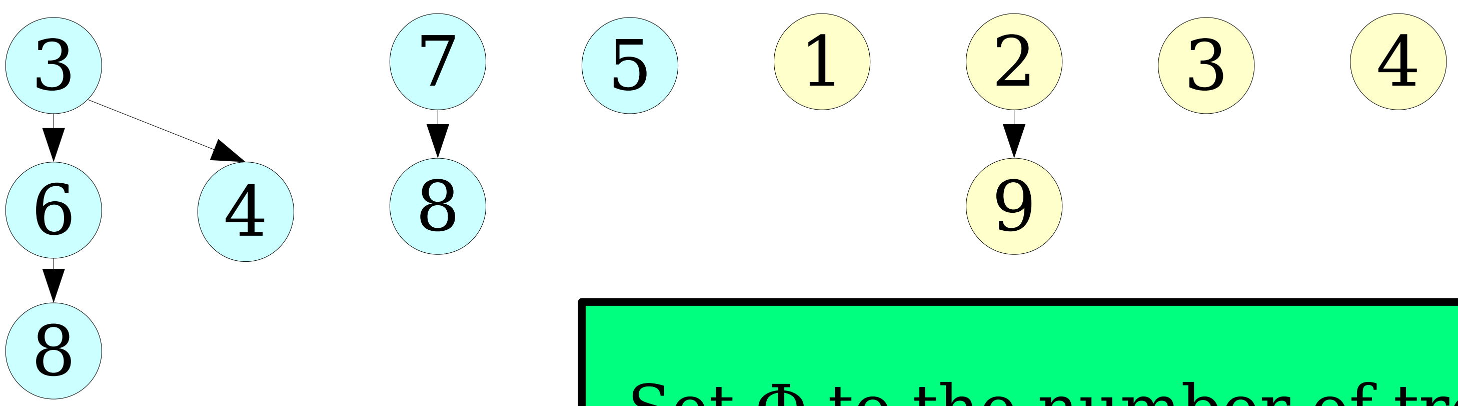
- What is the amortized cost of ***meld***?
- ***Common trick:*** When working with mergeable data structures, define Φ globally across all instances of the data structure.



Set Φ to the number of trees in the lazy binomial heap.

Analyzing a Meld

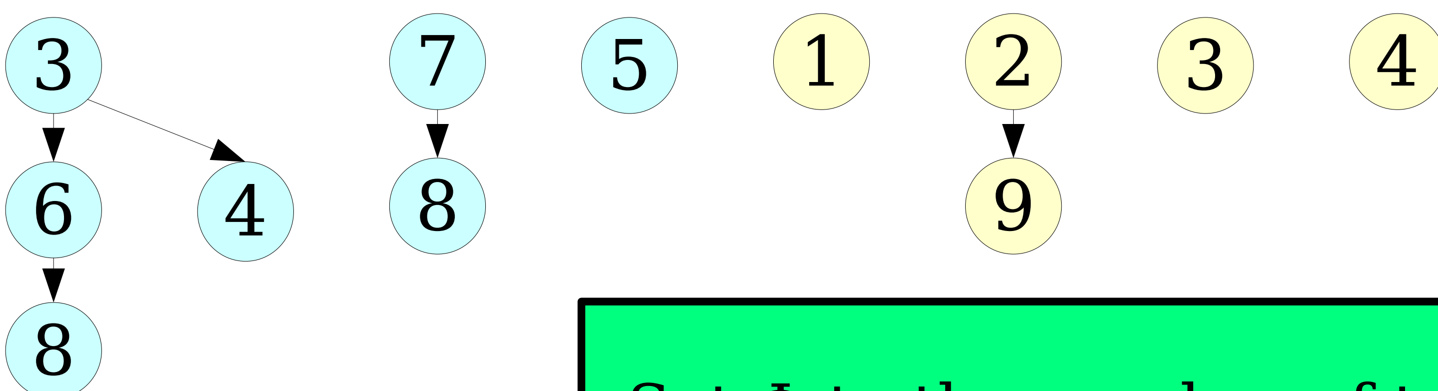
- What is the amortized cost of ***meld***?
- ***Common trick:*** When working with mergeable data structures, define Φ globally across all instances of the data structure.



Set Φ to the number of trees in ***all*** lazy binomial heaps.

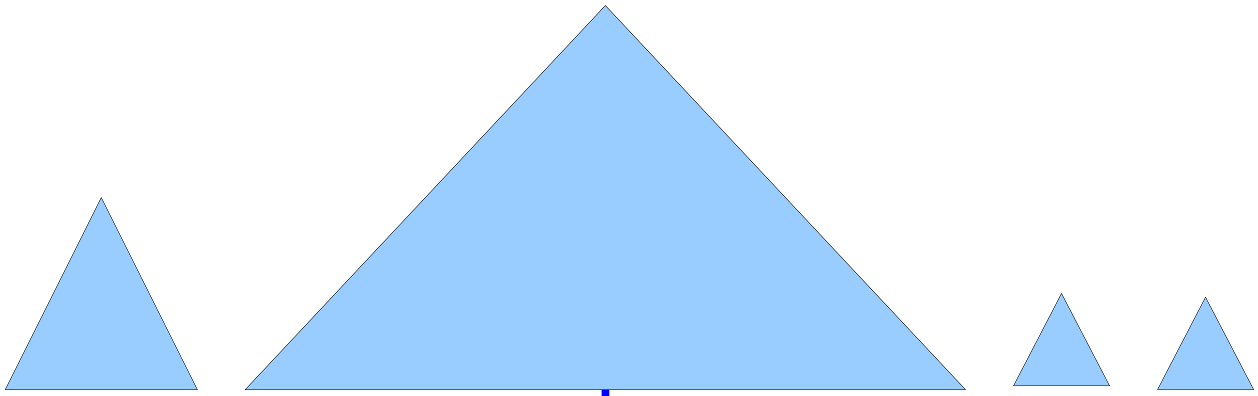
Analyzing a Meld

- What is the amortized cost of ***meld***?
- ***Common trick:*** When working with mergeable data structures, define Φ globally across all instances of the data structure.
- Now ΔΦ = 0 and the amortized cost is **O(1)**.



Set Φ to the number of trees in ***all*** lazy binomial heaps.

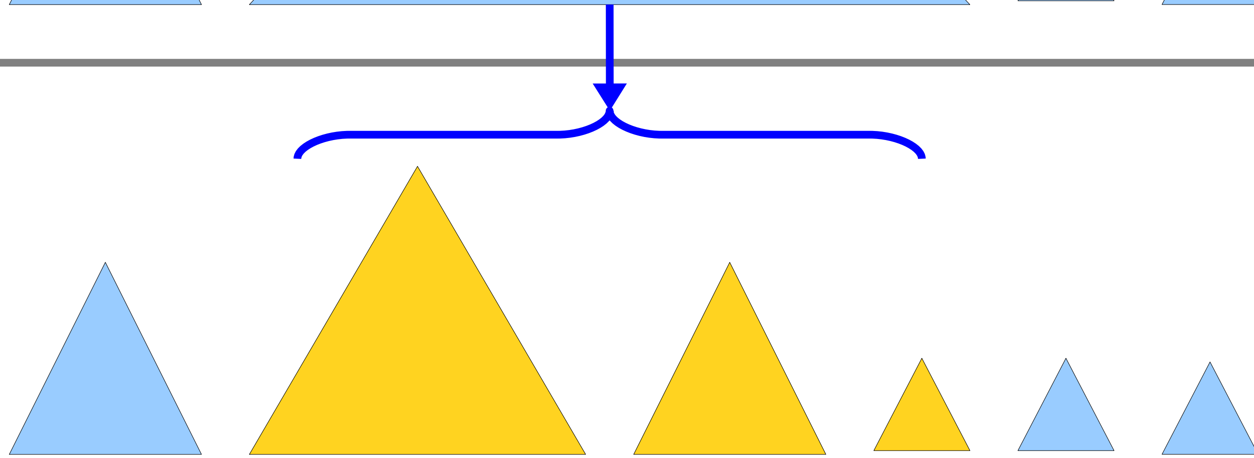
Analyzing ***extract-min***



Find tree with minimum key.

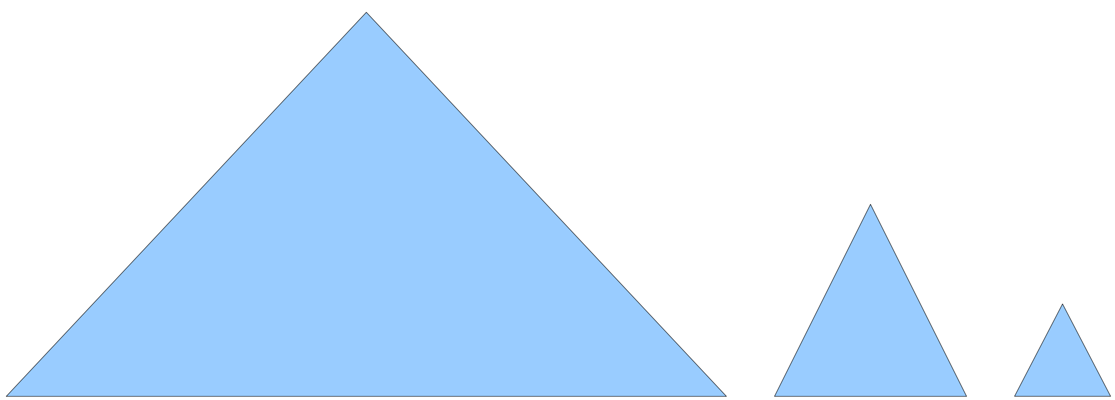
Work: O(t)

$\Phi = t$



Remove min. Add children to list of trees.

Work: O(log n)

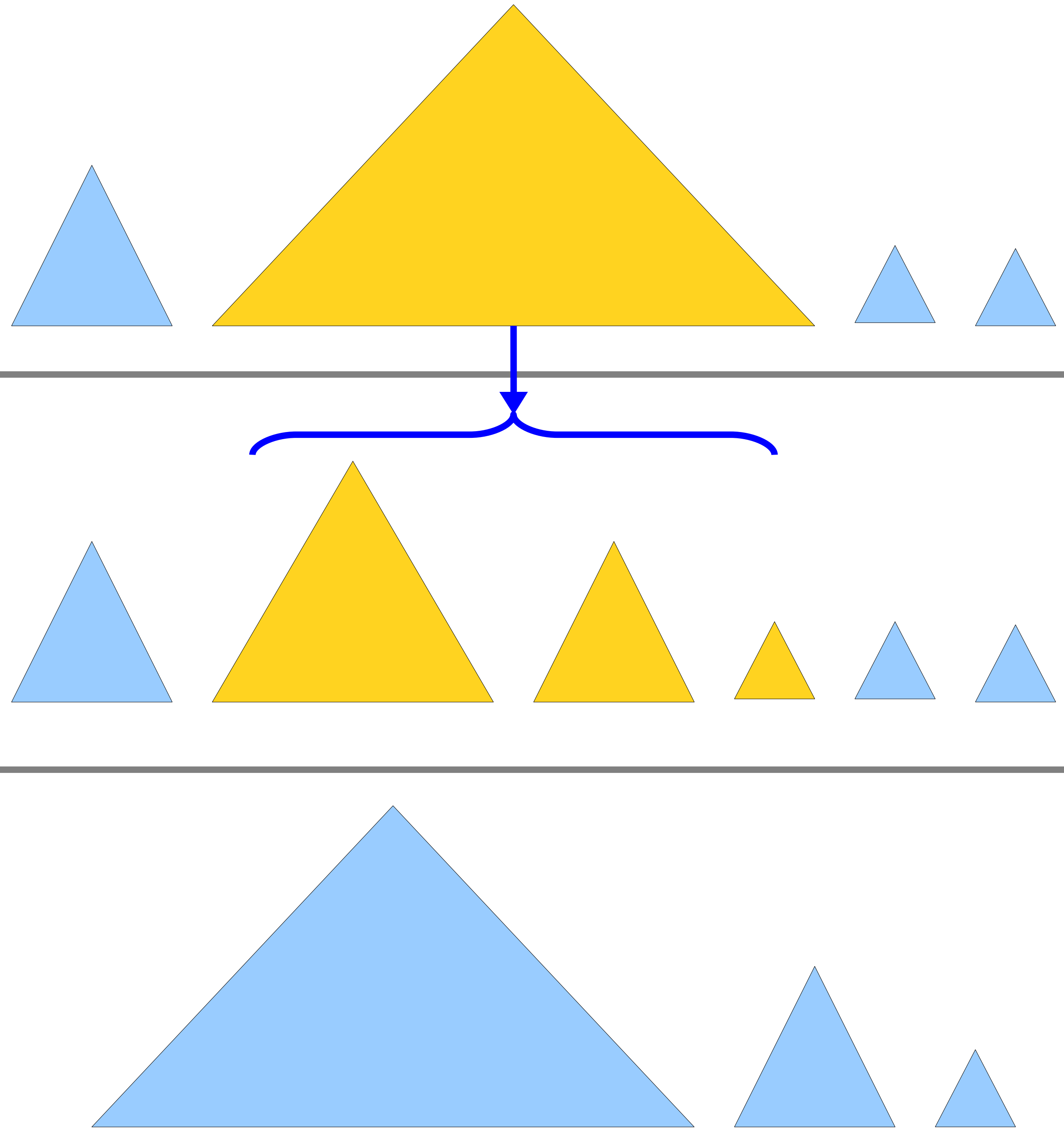


Run the coalesce algorithm.

Work: O(t + log n)

Φ = O(log n)

Work: **O(t + log n)** $\Delta\Phi$: **O(-t + log n)**



Find tree with minimum key.

Work: O(t)
$\Phi = t$

Remove min. Add children to list of trees.

Work: O(log n)

Run the coalesce algorithm.

Work: O(t + log n)
Φ = O(log n)

Amortized cost: **O(log *n*)**.

The Final Scorecard

- Here's the final scorecard for our lazy binomial heap.
- These are *great* runtimes! We can't improve upon this except by making ***extract-min*** worst-case efficient.
  - This is possible! Check out ***bootstrapped skew binomial heaps*** for details!

Lazy Binomial Heap

- ***Insert***: O(1)
- ***Find-Min***: O(1)
- ***Extract-Min***: O(log n)*
- ***Meld***: O(1)

* *amortized*

Major Ideas from Today

- Isometries are a *great* way to design data structures.
  - Here, binomial heaps come from binary arithmetic.
- Designing for amortized efficiency is about building up messes slowly and rapidly cleaning them up.
  - Each individual ***enqueue*** isn't too bad, and a single ***extract-min*** fixes all the prior problems.

Next Time

- ***The Need for decrease-key***
  - A powerful and versatile operation on priority queues.
- ***Fibonacci Heaps***
  - A variation on lazy binomial heaps with efficient ***decrease-key***.
- ***Analyzing Fibonacci Heaps***
  - A clever analysis.